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AN ANALYTICAL STUDY OF THE DYNAMICS OF AIRCRAFT IN UNSTEADY FLIGHT

By

H. C. Curtiss, Jr.

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FORT EUSTIS, VIRGINIA

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DEPARTMENT OF THE ARMY
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This report analyzes the influence of unsteady reference conditions on the dynamics of aircraft; particular emphasis is placed on the behavior of VTOL aircraft during transition.

Characteristics described in the analysis should be useful in explaining handling qualities of VTOL aircraft.

The report has been reviewed by this command and is considered to be technically sound.

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AN ANALYTICAL STUDY OF THE
DYNAMICS OF AIRCRAFT IN UNSTEADY FLIGHT

Princeton University Aerospace and Mechanical Sciences
Report No. 709

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for
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SUMMARY

The dynamic response of conventional and VTOL aircraft with varying flight velocity is investigated. It is assumed that the dynamic motions of aircraft may be described by linear differential equations whose coefficients (stability derivatives) are functions of flight velocity, and therefore vary with time. Primary emphasis is placed on the evaluation of the general nature of the vehicle response and its departure from frozen system (constant coefficients) characteristics.

An approximate solution to linear differential equations with variable coefficients is presented, which, roughly speaking, applies if the percentage change of each of the characteristic roots per unit time is small compared to the spacing of the frozen roots on the complex plane. This approximate solution is interpreted in terms of a distortion of the frozen locus of roots on the complex plane. Variable coefficient effects may be rapidly estimated using this result.

The properties of the solutions to linear differential equations with variable coefficients of significance to the problem are discussed. Of particular importance is the difference in the apparent damping of the various degrees of freedom in accelerated flight.

The short period motion of aircraft is examined in detail. For stability derivatives that vary linearly with velocity, typical of conventional fixed-wing aircraft, the variations in the time histories of pitching velocity and vertical velocity appear as a distortion of the time scale of the frozen response. The transients in angle of attack and normal acceleration will exhibit different damping characteristics. For an aircraft that is stable in steady flight, the angle of attack response may be unstable in decelerated flight, and the normal acceleration response may be unstable in accelerated flight.

The influence of other stability derivative variations on the short period motion and some examples of higher-order system dynamics are examined.

Simplifying assumptions are used whenever possible, such that the important features of the phenomena can be evaluated. Specific, detailed analyses can be carried out by using the method presented.

FOREWORD

This research was conducted by the Department of Aerospace and Mechanical Sciences, Princeton University, under the sponsorship of the United States Army Transportation Research Command, Fort Eustis, Virginia, under contract DA 44-177-AMC-8(T), Task Order No. 2. Lt. F. E. LaCasse was the USAAVLABS project officer.

The research was performed under the supervision of Professor Edward Seckel of the Department of Aerospace and Mechanical Sciences, Princeton University.

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LIST OF SYMBOLS

| | |
|-------------------|---|
| A | Propeller disc area, square feet; angle between body axis and wind axis, positive for body axis nose up with respect to wind axis; function of time |
| a, a_{ij} | Time-varying coefficients in differential equation; acceleration, feet per second per second |
| AR | Amplitude ratio; aspect ratio |
| a_w | Wing slope of the lift curve, per radian |
| b | Reference length of missile, feet; time-varying coefficient in differential equation |
| c | Wing chord, feet; damping coefficient; time-varying coefficient in differential equation |
| C_D | Drag coefficient $\left(\frac{D}{\frac{1}{2} \rho S V^2} \right)$ |
| C_i | Constant |
| C_L | Lift coefficient $\left(\frac{L}{\frac{1}{2} \rho S V^2} \right)$ |
| C_{L_α} | Rate of change of lift coefficient with angle of attack, per radian $\left(\frac{\partial C_L}{\partial \alpha} \right)$ |
| C_m | Pitching-moment coefficient $\left(\frac{M}{\frac{1}{2} \rho S c V^2} \right)$ |
| C_{m_q} | Rate of change of pitching-moment coefficient with pitching velocity, per radian per second $\left(\frac{\partial C_m}{\partial q} \right)$ |
| $C_{m_{\bar{q}}}$ | Rate of change of pitching-moment coefficient with non-dimensional pitching velocity $\left(\frac{\partial C_m}{\partial \bar{q}} \right)$ |

| | |
|------------------------|--|
| C_{m_u} | Rate of change of pitching-moment coefficient with horizontal velocity, per foot per second $\left(\frac{\partial C_m}{\partial u}\right)$ |
| C_{m_w} | Rate of change of pitching-moment coefficient with vertical velocity, per foot per second $\left(\frac{\partial C_m}{\partial w}\right)$ |
| $C_{m_{\dot{w}}}$ | Rate of change of pitching-moment coefficient with time rate of change of vertical velocity, per foot per second per second $\left(\frac{\partial C_m}{\partial \dot{w}}\right)$ |
| C_{m_α} | Rate of change of pitching-moment coefficient with angle of attack $\left(\frac{\partial C_m}{\partial \alpha}\right)$ |
| $C_{m_{\dot{\alpha}}}$ | Rate of change of pitching-moment coefficient with time rate of change of angle of attack, per radian per second $\left(\frac{\partial C_m}{\partial \dot{\alpha}}\right)$ |
| D | Drag, pounds |
| d | Time-varying coefficient in differential equation |
| E | Percentage error |
| e | Airplane efficiency factor; time-varying coefficient in differential equation; base of natural logarithms |
| f | Acceleration parameter $\left(\frac{\dot{U}}{wU}\right)$ |
| g | Acceleration due to gravity, feet per second per second |
| I | Vehicle moment of inertia, slug feet squared |
| i_w | Tilt angle of thrust producer of VTOL aircraft, radians |
| J_p | Bessel function of the first kind |
| k | Spring constant |

| | |
|---------------|---|
| k_y | Vehicle radius of gyration, feet $\sqrt{\frac{I}{m}}$ |
| L | Lift, pounds |
| l | Length of mathematical pendulum, feet |
| l_o | Wavelength of short period oscillation, feet $\left(\frac{U_o}{\omega_o}\right)$ |
| l_x | Distance between center of gravity and aerodynamic center of vehicle, positive for center of gravity ahead of aerodynamic center, feet |
| L_w | Rate of change of lift with vertical velocity, pounds per foot per second $\left(\frac{\partial L}{\partial w}\right)$ |
| M | Pitching moments divided by moment of inertia, positive nose up, per second squared |
| m | Vehicle mass, slugs; integer |
| M_q | Rate of change of pitching moment with pitching velocity, divided by moment of inertia, per second $\left(\frac{1}{I} \frac{\partial M}{\partial q}\right)$ |
| M_u | Rate of change of pitching moment with horizontal velocity, divided by moment of inertia, per foot-second $\left(\frac{1}{I} \frac{\partial M}{\partial u}\right)$ |
| M_w | Rate of change of pitching moment with vertical velocity, divided by moment of inertia, per foot-second $\left(\frac{1}{I} \frac{\partial M}{\partial w}\right)$ |
| $M_{\dot{w}}$ | Rate of change of pitching moment with time rate of change of vertical velocity, divided by moment of inertia, per foot-second squared $\left(\frac{1}{I} \frac{\partial M}{\partial \dot{w}}\right)$ |
| M_α | Rate of change of pitching moment with angle of attack, divided by moment of inertia, per second squared $\left(\frac{1}{I} \frac{\partial M}{\partial \alpha}\right)$ |

| | |
|--------------------|--|
| $M_{\dot{\alpha}}$ | Rate of change of pitching moment with time rate of change of angle of attack, divided by moment of inertia, per second $\left(\frac{1}{I} \frac{\partial M}{\partial \dot{\alpha}}\right)$ |
| M_{δ} | Rate of change of pitching moment with control deflection, divided by moment of inertia, per second squared $\left(\frac{1}{I} \frac{\partial M}{\partial \delta}\right)$ |
| N | Constant |
| n | Dummy variable |
| N_p | Bessel function of the second kind |
| N_z | Normal acceleration, g's |
| p | Period, seconds |
| Q | Dependent variable |
| q | Pitching velocity, positive nose up, radians per second; dynamic pressure, pounds per square foot |
| \bar{q} | Nondimensional pitching velocity $\left(\frac{qc}{U}\right)$ |
| R, r | Nondimensional parameter |
| r_i | Quasi-steady root, per second |
| S | Vehicle wing area, square feet |
| s | Dummy variable; independent variable |
| T | Thrust of propeller or engine, pounds |
| t | Time, seconds |
| U | Flight velocity, feet per second |
| u | Horizontal perturbation velocity, positive forward, feet per second |
| U_i | Solution to differential equation |

| | |
|------------|---|
| V | Flight velocity, feet per second |
| W | Vehicle weight, pounds; dependent variable |
| w | Vertical perturbation velocity, positive downward, feet per second |
| X | Horizontal force divided by mass, positive forward, per second squared |
| x, x_i | Dependent variables |
| X_q | Rate of change of horizontal force with pitching velocity, divided by mass, feet per second $\frac{1}{m} \frac{\partial X}{\partial q}$ |
| X_u | Rate of change of horizontal force with horizontal velocity, divided by mass, per second $\frac{1}{m} \frac{\partial X}{\partial u}$ |
| X_w | Rate of change of horizontal force with vertical velocity, divided by mass, per second $\frac{1}{m} \frac{\partial X}{\partial w}$ |
| y | Dependent variable |
| Z | Vertical force, divided by mass, positive down |
| Z_p | Linear combination of Bessel functions of first and second kind ($C_1 J_p + C_2 N_p$) |
| Z_u | Rate of change of vertical force with horizontal velocity, divided by mass, per second $\frac{1}{m} \frac{\partial Z}{\partial u}$ |
| Z_w | Rate of change of vertical force with vertical velocity, divided by mass, per second $\frac{1}{m} \frac{\partial Z}{\partial w}$ |
| Z_α | Rate of change of vertical force with angle of attack, divided by mass, per second squared $\frac{1}{m} \frac{\partial Z}{\partial \alpha}$ |
| α | Angle of attack, positive nose up, radians |

| | |
|--------------------------------|--|
| α_i | Constant |
| α_{o_L} | Angle of attack of zero lift, radians |
| β_i | Constant |
| γ | Flight path angle, positive for climb, radians |
| Δ | Perturbation |
| δ | Damping ratio; perturbation |
| δ_E | Vehicle pitching moment control deflection |
| δ_T | Vehicle power control deflection |
| ϵ | Fractional, nondimensional rate of change of a function of time |
| ζ | Nondimensional parameter; damping ratio |
| η | Nondimensional wing tilt rate, positive wing tilting down |
| θ | Vehicle pitch angle, positive nose up, radians; dependent variable |
| λ_i | Unsteady root, per second |
| μ | Vehicle density factor $\left(\frac{m}{\rho S c} \right)$ |
| ρ | Air density, slugs per cubic foot |
| σ | Damping coefficient, per second |
| $\tau, \bar{\tau}$ | Nondimensional time; time lag, seconds |
| $\tau_1, \bar{\tau}_1, \tau_2$ | Transformed independent variables |
| Φ_i | Constant; phase angle |
| ω | Frequency, radians per second |

SUBSCRIPTS

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|---------------|--|
| B | Quantity referred to body axis |
| E | Quantity evaluated at jet engine exit |
| e | Envelope of oscillation; effective value |
| f | Fuselage |
| g | Gravitational |
| i | Inertial |
| m | Reference value |
| o | Initial value |
| p | Quantity evaluated along a prescribed path |
| QS | Quasi-steady value |
| s | Evaluated in propeller slipstream |
| T | Tail |
| w | Quantity referred to wind axis |
| $\frac{1}{2}$ | Time to one-half amplitude |

OTHER

| | |
|-----------------------|--------------------------------------|
| \sim | Proportional to; order of |
| \cong | Approximately equal to |
| (\sim) | Average Value |
| $()^*$ | Characteristic value |
| $(\bar{})$ | Nondimensional parameter |
| (\wedge) | Specific value of parameter |
| $(\dot{})$ | Differentiation with respect to time |

CHAPTER I

INTRODUCTION

The classical approach to the prediction of the stability and control characteristics of aircraft is based on linearization of the differential equations of motion about a reference flight condition, steady flight. Steady flight refers normally to constant flight velocity and attitude, although the analysis may be extended to such quasi-steady situations as constant rolling velocity. The disturbed motion is then described by linear, constant coefficient differential equations. The linearity of the equations follows from the assumption that the deviations of the vehicle from the reference condition are infinitesimal, and the constant coefficient nature of the equations follows from the fact that the reference condition is steady, level flight. The tremendous advantage of this simplified mathematical model lies in the ability to obtain analytical solutions to the equations. As a result, a great many general conclusions regarding the relationships between the physical parameters of the vehicle, the flight conditions, and the nature of the disturbed motion may be drawn. Insight is gained that would have been obtained only with great difficulty by dealing directly with the "exact" nonlinear differential equations, even using automatic computation equipment.

Comparison of theory with experiment has proven this approach successful and applicable to the prediction of the response of aircraft to moderate control motions and other disturbances of such a magnitude to be of great practical value.

An extension of the classical approach is necessary to investigate the pitching and heaving motion of a ballistic missile about a zero lift trajectory. The differential equations describing this motion are linearized about the trajectory, in a similar fashion to the classical approach. Now, however, there are two factors that complicate the analysis. The reference condition, i. e., the trajectory, is no longer steady, and, in general, the vehicle velocity and flight path angle are changing with time. In addition, since the flight path is no longer necessarily horizontal, the air density is not constant.

The variation in these three quantities, with time, results in the description of the disturbed motion in terms of linear differential equations with variable coefficients. The analytical solution of these equations is not as simple as the classical case, and approximate solutions must be resorted to, in general. However, it is still possible to draw

useful conclusions about the motion without dealing with the complete non-linear equations.

The effect of density variation would appear in studying the dynamics of aircraft in climbing or diving steady flight and its importance would depend upon the flight speed of the airplane.

With the development of vertical takeoff aircraft, the ability to change flight velocity rapidly at slow flight speeds gives rise to similar novel questions in relation to the prediction of stability and response characteristics, and to the extension of the classical theory to include these effects.

In addition to the influence of the unsteady reference condition, i. e., changing flight velocity on the disturbed motion, two additional effects arise. The first concerns the possible effects of nonsteady aerodynamics becoming more important due to low flight speeds, and the second arises from the fact that a rapidly accelerating VTOL airplane at slow flight speeds will experience somewhat different flight conditions than encountered in steady flight. For example, a tilt-wing airplane accelerating rapidly from hover to forward flight will experience a wing tilt angle, velocity relationship different from steady flight at corresponding velocities. This effect arises from the fact that the weight of the airplane is in part supported by propeller thrust. This effect would not be significant in conventional airplane flight where the engine thrust is essentially horizontal so that the condition of level flight determines the relationship between flight velocity and angle of attack.

The investigation here is primarily concerned with the first question, which is: What is the influence of nonsteady flight conditions on the dynamics of airplanes? The latter questions are considered briefly; however, particularly in the case of the second effect, more experimental data are needed to evaluate the importance of these effects.

We wish, then, to primarily study the dynamics of aircraft in non-steady flight with particular reference to VTOL aircraft, where it would be expected that the importance of nonsteady effects would be greatest for the following reasons:

First, the ability to generate large accelerations and decelerations along the flight path at slow flight speeds would give rise to large percentage changes in velocity in a comparatively short time.

Second, rather large changes in the nature of the disturbed motions at various steady flight conditions are obtained from the classical approach.

The problem will be considered primarily with the view of obtaining exact or approximate analytical solutions to the problem such that general conclusions concerning the nature of these effects may be drawn.

The introduction of the nonsteady reference condition, as mentioned, changes the mathematical model from linear constant coefficient differential equations to linear variable coefficient equations and so we expect more difficulty in obtaining "exact" solutions to the problem.

The particular questions we attempt to answer in this study are:

1. What parameters are important in determining the departure of the description of the motion from the classical steady flight description?
2. What is the nature of the actual disturbed motion of the vehicle in nonsteady flight and how is it best described?
3. Are there any unusual phenomena that may occur due to nonsteady flight that would not be foreseen on the basis of the simpler classical approach at neighboring steady flight conditions?

The following plan, then, is followed with respect to the approach to this problem.

First we discuss pertinent literature in the field, in particular the analysis of missile dynamics, which bears a close mathematical relationship to the problem under consideration.

Then, since we will be dealing with linear equations with variable coefficients, we present a discussion of the nature of the solutions to these equations using as a reference point the characteristics of constant coefficient equations. Most of the classical aspects of the investigation of equations with variable coefficients deal with second order equations. Since in many problems associated with the disturbed motion of aircraft we must deal with differential equations of higher order, a new method of approach to the approximate solution of higher order equations with varying coefficients is presented.

We then proceed directly to the problem at hand, starting with the simplest case, that of an airplane with conventional stability derivatives in nonsteady flight, and then characteristics typical of VTOL aircraft. The analysis of the motions is limited to the case of responses to initial conditions, since it is considered that this problem should be well understood before proceeding to the more complex case of response to control.

An attempt is made to provide a complete description of all aspects of the disturbed motion as influenced by nonsteady flight, since this description is not found in the literature. A physical description of the pertinent phenomena is given.

The results are presented in nondimensional form suited to the problem such that the importance of unsteady effects can be estimated from the nature of calculations made on the basis of the classical steady flight approach in the neighborhood of the unsteady flight conditions and the nature of the unsteady flight conditions.

We will only consider briefly the nature of the unsteady reference motions, and will not consider other problems, such as possible difficulties in trimming the airplane that may also be of significance in accelerated flight. That is, we consider the counterpart of stick-fixed motion in unsteady flight.

The emphasis of this study is to obtain an estimate of the primary factors influencing the phenomena of interest. Simplifying assumptions are used whenever it is considered that they will not influence the basic nature of the results.

CHAPTER II

PREVIOUS INVESTIGATIONS ON RELATED STABILITY PROBLEMS

There are three areas of research related to the problem under consideration. The first is the direct problem of the effects of unsteady flight on the dynamics of aircraft (References 1 and 2). Similar problems have been investigated with respect to aeroelastic systems (References 10, 11, and 21). The second, a related area which has received considerable attention in recent years, is the analysis of the dynamic stability of missiles entering or leaving the atmosphere (References 3, 4, 5, 6, 7, 8, and 9). The third area relates directly to the mathematics involved in the problem; that is, the solution of linear differential equations with variable coefficients. The literature in this field is very large, and so we restrict the discussion to those results pertinent to our specific problem and do not attempt a comprehensive review of the subject. A very complete bibliography of engineering literature on this subject is given in Reference 12.

It should be noted at the outset that we will only consider coefficients that vary roughly in a monotonic fashion with time. We will not consider equations with periodic coefficients, since these are not typical of the variation of the coefficients of the differential equations describing the dynamic motions of an aircraft in unsteady flight.

Perhaps the most annoying aspect of equations with time varying coefficients is that their solutions, in general, cannot be expressed analytically in terms of simple functions (Reference 32). Therefore, if we wish to obtain results in an analytical form, we must resort to approximate methods. The most powerful idea in relation to approximate solutions is that of an asymptotic solution (Reference 16); i. e., the solution that becomes valid as time, the independent variable, in some sense becomes very large (Reference 22). This may be real time becoming large or some transformed independent variable becoming large. Exactly what function of the independent variable must be large will depend upon the nature of the coefficient variation. As a simple example, if the coefficients of the differential equations are asymptotic to constants, it might be expected that the asymptotic solution would be the solution of the constant coefficient equation that results from neglecting the varying parts of the coefficients. While we wish to consider differential equations with more general coefficient variations than this, in many instances it is possible to transform a given differential equation into a form where the coefficients are asymptotic to constants. Then with suitable restrictions

on the varying part of the coefficients, an asymptotic solution will be valid (References 16 and 23).

Also, one finds in the physics literature an approximate solution to a second-order differential equation referred to as the WKB solution (References 12 and 20) or in some cases as the Liouville-Green approximation (Reference 30), after its original authors.

This approximation is the asymptotic solution to a second-order differential equation with certain restrictions on the coefficients. It has been rederived by many authors, particularly in the subject of our investigation (References 2 and 3). We will discuss this approach in some detail in Chapter III.

The above results, except in special cases, are applicable to second order equations, and there has been little investigation of higher order differential equations.

An interesting approach to approximate solutions of second order equations with time varying coefficients using an analogue computer is given in Reference 28. The authors consider approximating a time varying system by an equivalent constant coefficient system. The coefficients of the equivalent constant coefficient system are obtained by minimizing the mean square of the differences in the displacement response. It was found that one cycle of the oscillation was necessary to obtain a reasonably sensitive result. The procedure worked satisfactorily for coefficients varying linearly with time if the fractional changes in the coefficients per unit time was restricted. The range of coefficient variations studied can be approximated reasonably well by asymptotic solutions. This approach was limited to coefficients that varied by a factor of 4 in one initial cycle. Faster variations did not give satisfactory results for reasons discussed later.

For completeness, these investigations are discussed in some detail in Chapter III.

In relation to the specific problem of the dynamics of aircraft in unsteady flight, the following papers are pertinent.

In Reference 1 the longitudinal, short period dynamic stability of an aircraft in accelerating and decelerating level flight has been studied. The author was primarily interested in developing stability criteria for the short period motion rather than integrating the differential equations. Essentially, the second method of Liapounov (Reference 13) is applied to derive stability criteria for the angle of attack and pitching velocity

motion. The conclusions of this article are that a statically stable airplane is also dynamically stable in the short period phase of the disturbed motion if the basic straight level flight is accelerated, and, for decelerating flight, that the stability improves with increase of flight speed and it gets worse when the absolute value of the acceleration is increasing, leading to the possibility of instability. The stability criteria for deceleration given are:

$$\frac{1}{g} \left| \frac{dV}{dt} \right| < \frac{C_{L\alpha} + C_D}{C_L}$$

$$\frac{1}{g} \left| \frac{dV}{dt} \right| < - \frac{1}{C_L} \left(\frac{V}{V_m} \right) (C_{m_q} + C_{m_{\dot{\alpha}}}) .$$

These expressions are sufficient conditions for stability, but they are very conservative. It is only necessary to require that:

$$\frac{1}{g} \left| \frac{dV}{dt} \right| < \frac{C_{L\alpha} + C_D}{C_L} - \frac{1}{C_L} \left(\frac{V}{V_m} \right) (C_{m_q} + C_{m_{\dot{\alpha}}}) .$$

Some estimate is made of the difference in damping of the two variables, but no definite conclusions are drawn. We will show in Chapter V that only the angle of attack response can be unstable when the airplane is stable, and that the pitching velocity response is always stable when the airplane is stable.

The equations of motion employed in this reference are identical to those presented here, and thus are exactly integrable if the correct dependence of the stability derivatives on velocity is assumed. The manner in which the stability derivatives are expressed in this reference implies that C_{m_q} and $C_{m_{\dot{\alpha}}}$ are constant rather than being inversely proportional to speed, the usual variation for a conventional airplane (Reference 14).

Reference 2 also evaluates the short period dynamic stability of an airplane in accelerating and decelerating level flight. For a second order differential equation, the author derives the WKBJ solution (References 12 and 20) and notes that it is an asymptotic solution. He then develops solutions by a similar approach for fourth-order, two-degree-of-freedom systems. The results are incorrect, however, due to the invalid assumption that each of the dependent variables is a linear combination of the

same functions of the dependent variable, time. This aspect of time-varying equations is discussed in Chapter III.

The author discusses some interesting concepts with regard to stability criteria of systems in general, and then applies the WKBJ approximation to the short period motion, considering angle of attack motion with constant acceleration and deceleration. A term $\frac{V}{V}$ is neglected in the damping coefficient of the differential equation, resulting in an error in the solution for angle of attack response. The denominator term in equation 4.4 should be raised to the first power and not the one-half power. While the term neglected is small, it is the same order as the other effects present, so to obtain consistent results this term should be retained. In the latter part of this chapter we consider this effect with respect to missile dynamics, where it may cause considerable confusion.

An estimate is made of the magnitude of acceleration and deceleration that will result in a 10-percent change in the time to 1/2 amplitude of the angle of attack response. It is concluded that the magnitude of the change for current aircraft is barely noticeable.

Again, in this paper, approximate methods are used to treat the exactly integrable case of level accelerated and decelerated flight, as conventional stability derivatives are used.

Reference 10 discusses in some detail the differences between the time histories of the displacement, velocity, and acceleration responses of a second order system with linearly varying coefficients. As an example, a second order differential equation with linearly varying coefficients is solved approximately in terms of Bessel functions, and asymptotic expansions of the Bessel functions are used to obtain simple expressions for the envelope of the displacement, velocity, and acceleration responses. Implications of the results on flutter testing under accelerated and decelerated conditions are discussed. In particular, the difference in growth rate of the displacement compared to the velocity and acceleration is noted.

Reference 11 contains some interesting comments on time-varying systems and the problems associated with defining stability in a time-varying system. A preliminary investigation of higher-order systems is made by assuming a specific coefficient variation that results in Euler's differential equation¹⁶. There is also a rather inconclusive approach to the lateral motion of the airplane. The conclusions of this article are that the acceleration effects depend on a dimensionless number, $\frac{1}{V^2} \frac{dV}{dt} C$,

and that to obtain changes in a critical flutter speed, the stiffness of the system must be varying, and that variable damping does not alter a critical flutter speed.

We have discussed research related to the influence of variable flight velocity on airplane dynamics, and now we describe some results concerned with the dynamic stability of missiles entering or leaving the atmosphere.

References 3, 4, 5, 6, 7, 8, and 9 are dynamic stability analyses of missiles. We are primarily concerned with the assumptions and nature of the solutions, and not with the significance of the results as they relate to the design of missiles.

Reference 7 considers a spinning missile, and is not of direct interest.

References 3, 4, 5, 6, 8, and 9 evaluate the dynamic stability of a missile entering or leaving the atmosphere. Reference 6 considers a spinning missile as well as a nonspinning missile. We will restrict our discussion on Reference 6 to the nonspinning case only, as pertinent to the problem under study.

The standard approach to the problem has been to separate the two-degree-of-freedom short-period motion from the trajectory equations and then to investigate the angle of attack time history. References 3, 6, 8, and 9 use identical angle of attack equations for a missile with no thrust. Only the homogeneous solution of the equations is discussed. The stability derivatives for a missile are assumed to vary with flight velocity in a manner referred to in Chapter V as conventional airplane. In these references this variation is assumed to be valid for hypersonic flight.

In Reference 3, an expression for the envelope of the oscillatory motion is presented, using the WKBJ approximation which is derived from an energy approach. A complicated expression for the frequency is obtained by approximating the spring constant over small portions of a cycle. As noted in Reference 9, and in Chapter III here, this treatment of the frequency is an unnecessary refinement by comparison to the approximation for the amplitude of the motion. Later in the paper, the authors refer to the kinetic energy of the motion being constant. This is not, in general, true in a time-varying system as discussed in Appendix IV. In addition, the kinetic energy of a missile is not determined by the mass times the square of the rate of change of angle of attack with time. This paper does clearly indicate the differences in the time

histories of angle of attack, rate of change of angle of attack with time, and the second derivative of angle of attack with time.

In Reference 6, the solution to the angle of attack equation is obtained also by use of the WKBJ approximation. A method of successive approximation is also presented to refine the solution. The author of Reference 6 states that his results differ from References 3 and 8. The difference between the results obtained in these three papers is due to the nature of the trajectory equation used in rearranging the expression for the angle of attack response as discussed at the end of this chapter. The expression for the angle of attack response contained in Reference 6 contains an additional term, due to the iterative procedure developed. This term must be small by the nature of the iteration. The author also presents an approximate solution to a two-degree-of-freedom, fourth-order system by using an extension of the WKBJ approach. A special case is investigated where the equations of motion are symmetric, and so the two variables must involve the same functions of time, as distinguished from an arbitrary system such as that considered in Reference 2 where the variables, in general, will involve different functions of time.

References 8 and 9 present similar analyses for the angle of attack motion, also studying in detail the motion and the importance of various terms. The independent variable, time, is transformed to distance traveled (altitude), thus eliminating the dependence of the coefficients on velocity. The coefficients of the transformed differential equations vary only due to density, and are solved using Bessel functions. There appears to be confusion in Reference 8, where the first derivative of angle of attack with respect to time is referred to as pitching velocity and the second derivative is referred to as angular acceleration. A two-degree-of-freedom system is considered, and since the system is time varying, the time history of pitching velocity with time will be different from the time history of the rate of change of angle of attack with time as discussed in Chapter V. Reference 9 presents considerable discussion on the separation of the trajectory equations from the oscillatory equations, arguing that the former terms are roughly constants and that the latter are oscillatory terms, so that each must be separately equal to zero. This question is taken up in Chapter V.

Reference 4 considers also the motion of a missile from launching. Body axes are used to derive the equations of motion. Thrust is included in the equations of motion; air density and acceleration along the flight path are assumed constant. The differential equation for angle of attack is investigated. The author studies forced motion, as well as the unforced motion. The homogeneous angle of attack equation is equivalent to the equations discussed above. The method of solution is to transform

the independent variable from time to distance traveled and then solutions are obtained in terms of Bessel functions of order $\pm 1/2$; i. e., the assumption of constant density results in a simple form for the solutions. The pitch angle response is then computed from the angle of attack response. The forced analysis becomes quite lengthy. Two sample responses are presented, but there is no discussion of the results. Graphs of functions are presented to aid in evaluating the nonhomogeneous response.

Reference 5 then sets out to show the equivalence of the differential equation for angle of attack presented in Reference 4 compared to those in References 3, 6, 8, and 9, since the method of derivation makes them appear slightly different. The author arrives at the rather surprising and erroneous conclusion that different results are obtained depending on the axis system and the variables used. This, of course, must be due to the fact that different assumptions were used in the derivation of the equations. One inconsistency is the fact that the wind axis equations from References 3, 6, 8, and 9 are valid only when there is no thrust, while the equation given in Reference 4 is valid when thrust is present, since body axes are employed. The inconsistency appears when the coefficients in the differential equations are compared on the basis of high drag and low deceleration, a situation not physically possible without thrust. This difference is explained fully in Appendix II. Reference 5 also notes that, using the equations of References 3, 6, 8, and 9, and transforming the independent variable from time to distance traveled, the equation of motion will be a constant coefficient equation if density is constant. The nature of the amplitude of the angle of attack motion is discussed briefly.

To summarize these papers we will discuss briefly the results. The angle of attack equation presented in References 3, 6, 8, and 9 is:

$$\frac{d^2\alpha}{dt^2} + \left(C_{L_\alpha} \frac{S}{m} - C_{m_q} \frac{Sb^2}{I} \right) \frac{q}{V} \frac{d\alpha}{dt} + \left(- C_{m_\alpha} \frac{bS}{I} \right) q \alpha = 0. \quad (\text{II-1})$$

This equation, precisely speaking, is restricted to level flight with no thrust, and small terms in the restoring force are neglected. This is equation 13 in Reference 3, using approximation 52, equation 52, in

Reference 6, where C_{m_α} is included and $C_{m_\alpha} = - (C_{L_\alpha} + C_D) \frac{l_x}{b}$; equation 6 in Reference 9, with assumptions made later to reduce it to this

form; and equation 8 in Reference 5. Equation 16 of Reference 4 does not appear identical, since thrust is included and density is assumed constant, but will reduce to equation II-1 with the assumption of no thrust.

The nature of the angle of attack oscillations is determined by removing the damping term with the transformation

$$\alpha_1 = \alpha_e \left(\frac{1}{2} \int_0^t \left[C_{L\alpha} \frac{S}{m} - C_{mq} \frac{Sb^2}{I} \right] \frac{q}{V} ds \right) \quad (\text{II-2})$$

The contribution of this transformation to the restoring force term is neglected (see Chapter III), such that the equation for α_1 is:

$$\frac{d^2 \alpha_1}{dt^2} + \left[-C_{m\alpha} \frac{bS}{I} \right] q \alpha_1 = 0. \quad (\text{II-3})$$

Then solving this equation by the WKBJ method (Reference 20), the expression for the envelope of the angle of attack motion will be:

$$\alpha_e = \alpha_o \left\{ \frac{1}{\left(-C_{m\alpha} \frac{bS}{I} \right)^{\frac{1}{4}} q^{\frac{1}{4}}} \right\} e^{-\frac{1}{2} \int_0^t \left[C_{L\alpha} \frac{S}{m} - \frac{C_{mq} S b^2}{I} \right] \frac{q}{V} ds}, \quad (\text{II-4})$$

or since $(C_{m\alpha} \frac{bS}{I})$ is a constant,

$$\alpha_e \sim \frac{1}{q^{\frac{1}{4}}} e^{-\frac{1}{2} \int_0^t \left[C_{L\alpha} \frac{S}{m} - C_{mq} \frac{Sb^2}{I} \right] \frac{q}{V} ds} \quad (\text{II-5})$$

This is equation 61 in Reference 9, equation 60 in Reference 6, (the +1 in the numerator appears due to the iteration used), and equation 54 in Reference 3. To obtain the form given in Reference 9, one must use the trajectory relationship,

$$m\dot{V} = -\frac{C_D \rho S V^2}{2},$$

which assumes that gravity is not important.

Integrating this relationship, we obtain

$$\frac{1}{\sqrt{V}} = e^{\frac{1}{2} \int_0^t C_D \frac{S}{m} \frac{q}{V} ds} \quad (\text{II-6})$$

and therefore

$$\alpha_e \sim \frac{1}{\rho^{\frac{1}{4}}} e^{\frac{1}{2} \int_0^t [C_D \frac{S}{m} - C_{L\alpha} \frac{S}{m} + C_{mq} \frac{Sb^2}{I}] \frac{q}{V} ds} \quad (\text{II-7})$$

This is the result presented in Reference 8, equation 30.

Equation II-5 is easier to interpret, does not depend on the trajectory equations, and is perhaps less confusing than II-7. Note that the relationship used to obtain II-7 assumes that gravity is not important in determining the trajectory, as pointed out in Reference 6.

Thus, the basic conclusion of these studies is that the angle of attack envelope is influenced by the aerodynamic damping as well as the dynamic pressure variation with time. It is concluded that even with no natural damping of the airframe, the amplitude of the angle of attack will decrease with time whenever the dynamic pressure is increasing with time. Or in other words, whenever the spring constant is increasing with time.

There is little consideration of the conditions necessary for application of approximate solutions (asymptotic solutions are not valid when the missile is at the "edge" of the atmosphere), or how different variations in the stability derivatives with velocity will affect the results. Also there is little discussion of the nature of the time histories of other variables of interest in the problem. We shall consider these questions in detail.

CHAPTER III

LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

The properties of the solutions of linear differential equations have been the subject of many investigations. We have not attempted to review this large field in Chapter II, and in this chapter we will only discuss those results pertinent to our particular problem. Extensive bibliographies are given in References 12 and 36. The nature of the nonhomogeneous (forced) solutions will not be considered. Only homogeneous (transient) solutions are discussed.

We begin this chapter with a discussion of some general properties of the solutions to linear differential equations, paying particular attention to those properties that are a departure from the familiar properties of constant coefficient differential equations. We take this point of view because of the nature of the problem of interest. We wish to consider the dynamic motions of an aircraft as it passes continuously through a series of flight conditions. When it is in equilibrium flight at any one of these flight conditions, its dynamic motions may be described by linear constant coefficient differential equations. Thus, we would expect that the form of the results in the unsteady case would be similar to the steady, constant coefficient case. We would like to be able to state our results in terms of the deviations of the motions in unsteady flight from the classical steady flight results. We consider some of the properties of the solutions to a second-order linear differential equation, interpreting the results in physical terms.

Since, in the problem of interest, as in many engineering problems, we are faced with systems of a higher order than second, in Chapter IV we develop a new method for the approximate solution of higher-order equations, and a convenient way of interpreting the results.

A. General Considerations

A linear second-order equation with variable coefficients may be written as

$$\frac{d^2 x}{dt^2} + c(t) \frac{dx}{dt} + k(t) x = 0, \quad (\text{III-1})$$

where in terms of a mechanical analogy we may think of $c(t)$ as a viscous damping term that varies with time, $k(t)$ as a spring or restoring force

that varies with time, and x as a displacement. In the following we use this nomenclature, referring to the coefficient of x as restoring force, and the coefficient of $\frac{dx}{dt}$ as damping, even though we may have no specific physical system in mind.

Two transformations are useful in considering the nature of the solutions to equation III-1.

The first is a transformation of the dependent variable that removes the terms proportional to velocity (damping term).

Let
$$\theta(t) = x(t) e^{\frac{1}{2} \int^t c \, dn} \quad (\text{III-2})$$

The differential equation in terms of θ is:

$$\frac{d^2\theta}{dt^2} + \left(k(t) - \left(\frac{c(t)}{2} \right)^2 - \frac{1}{2} \frac{dc}{dt} \right) \theta = 0. \quad (\text{III-3})$$

This transformation preserves the zeros of the original variable and adds two terms to the restoring force. The first, $\left(\frac{c}{2} \right)^2$, is the usual effect of damping on frequency, and the second, $\left(\frac{1}{2} \frac{dc}{dt} \right)$, is new, arising from the time-varying nature of the damping.

The transformation indicates that we may consider

$$\frac{d^2\theta}{dt^2} + b(t) \theta = 0 \quad (\text{III-4})$$

as the canonical form of a second-order linear time-varying equation.

A second transformation, applied to the independent variable, results in a constant restoring force term.

Let
$$s = \int_0^t k^{\frac{1}{2}}(n) \, dn. \quad (\text{III-5})$$

Equation III-1 becomes:

$$\frac{d^2x}{ds^2} + \left(\frac{c(t)}{\sqrt{k(t)}} + \frac{1}{2} \frac{1}{k^{\frac{3}{2}}} \frac{dk}{dt} \right) \frac{dx}{ds} + x = 0 \quad (\text{III-6a})$$

or

$$\frac{d^2x}{ds^2} + \left(\frac{c(s)}{\sqrt{k(s)}} + \frac{1}{2} \frac{1}{k(s)} \frac{dk}{ds} \right) \frac{dx}{ds} + x = 0. \quad (\text{III-6b})$$

If we apply the dependent variable transformation, III-2, to equation III-6b, denoting

$$2\zeta' = \frac{c}{\sqrt{k}} + \frac{1}{2} \frac{1}{k} \frac{dk}{ds}, \quad (\text{III-7})$$

and letting $\theta = x e^{\int_0^s \zeta'(n) dn}$, (III-2)

we obtain $\frac{d^2\theta}{ds^2} + (1 - (\zeta')^2 - \frac{d\zeta'}{ds}) \theta = 0.$ (III-8)

Equation III-8 is of the form anticipated in Chapter II, where the only coefficient in the differential equation is a constant plus a varying term.

We may expect that if ζ' and its rate of change are sufficiently small compared to 1, an approximate solution to the differential equation III-4 may be obtained by neglecting the latter two terms in equation III-8. Equation III-8 becomes a constant coefficient equation, and is readily solved. We anticipate our results and refer to equation III-7 as an apparent damping. If this apparent damping is small, it appears that we can find an approximate solution. These implications will be investigated in detail in succeeding sections.

The limitation of an approximate solution to lightly damped systems is a particularly interesting one, since it indicates that many situations of practical interest can be studied using approximate methods; that is, cases on the boundary of stability and instability.

Now, let us consider some general properties of the solutions to equation III-4, and then return to the question of approximate solutions.

It is natural to think of equation III-4 in terms of the deviations of the behavior of its solutions when the restoring force is a function of time from the case where the restoring force is a constant and the solutions are well known and easily manipulated. This viewpoint is particularly useful in relation to the physical problem of interest here.

Unfortunately, the exact solutions of the equation III-4 when the restoring force is some arbitrary function of time usually cannot be expressed in terms of well-known functions, or at least in terms of functions that can be easily manipulated and whose general behavior can be simply interpreted.

The theorems of Sturm (Reference 16) are helpful in indicating that our intuition and knowledge of constant coefficient equations can be extended to equations with varying coefficients, and so we list them here.

If a solution has no more than one zero in an interval, it is said to be nonoscillatory.

Given the differential equation III-4,

$$\frac{d^2\theta}{dt^2} + b(t) \theta = 0 \quad \text{with} \quad \begin{aligned} \theta(t_0) &= \alpha_1, \\ \frac{d\theta}{dt}(t_0) &= \alpha_2. \end{aligned} \quad (\text{III-4})$$

The theorems of Sturm are:

1. The zeros of two real linearly-distinct solutions of a differential equation of second order separate one another; i. e., the zeros of all solutions of a given differential equation oscillate equally rapidly.
2. A sufficient condition that the solution of the differential equation III-4 should have at least m zeros in the interval (t_0, t_1) is that $b(t) \geq \frac{m^2 \pi^2}{(t_1 - t_0)^2}$. If $b(t)$ is considered as the instantaneous value of the frequency squared, then the instantaneous period of the motion must always be equal to or less than $\frac{2(t_1 - t_0)}{m}$; i. e., equal to or less than the constant period that would result in m zeros in the interval.
3. If $b(t) \leq 0$ throughout an interval (t_0, t_1) the solutions of equation III-4 are nonoscillatory in this interval; i. e., if the restoring force gradient is negative, the system does not oscillate.
4. Given two differential equations,

$$\theta_1(t_0) = \alpha_1 ,$$

$$\frac{d^2\theta_1}{dt^2} + b_1 \theta_1 = 0 \quad \text{with} \quad (III-9a)$$

$$\frac{d\theta_1}{dt}(t_0) = \alpha_2 ,$$

$$\theta_2(t_0) = \beta_1 ,$$

$$\frac{d^2\theta_2}{dt^2} + b_2 \theta_2 = 0 \quad \text{with} \quad (III-9b)$$

$$\frac{d\theta_2}{dt}(t_0) = \beta_2 ,$$

and throughout the interval (t_0, t_1) , $b_2 \geq b_1$.

Also,

- a. α_1 and α_2 are not both zero, nor are β_1 and β_2 . This restriction eliminates the possibility of the trivial solution.
- b. If $\alpha_1 \neq 0$, then $\beta_1 \neq 0$. If $\theta_1(t)$ has m zeros in the interval $t_0 \leq t \leq t_1$, then $\theta_2(t)$ has at least m zeros in the same interval, and the i^{th} zero of $\theta_2(t)$ is less than the i^{th} zero of $\theta_1(t)$.

From these theorems it may be seen that the following properties of solutions to linear variable coefficient equations are similar to the behavior of solutions to linear constant coefficient equations:

From 1, roughly speaking, both solutions of a second-order differential equation with oscillatory solutions have the same frequency.

From 2, the minimum value of the spring constant over an interval gives a lower limit on the number of zeros (frequency) in an interval.

From 3, when the restoring force gradient is negative, the system is nonoscillatory.

From 4, as the average value of the restoring force gradient increases, the number of zeros, i.e., the frequency, increases. In other words, $b(t)$ still is roughly the square of an instantaneous frequency even when $b(t)$ is varying.

These similarities between constant coefficient and variable coefficient equations result in comparison theorems (Reference 16).

The solutions of variable coefficient equations differ in one important respect from the solutions of constant coefficient equations that is of considerable significance relative to the dynamic characteristics of physical systems.

For the homogeneous solution to a system of constant coefficient differential equations, all dependent variables and their derivatives may be expressed in terms of the same functions of time. This is no longer true in a variable coefficient system^{12, 20} In a single-degree-of-freedom system, the dependent variable and its derivatives will, in general, exhibit different characteristics; e. g., the amplitude of the displacement may decrease with time while the amplitude of the velocity may increase with time. In a multiple-degree-of-freedom system, different variables may exhibit different characteristics; that is, the amplitude of one variable may decrease with time while the amplitude of another variable may increase with time.

These properties can be shown by the following considerations:

Given the differential equations

$$\frac{dx_1}{dt} = a_{11}(t) x_1 + a_{12}(t) x_2, \quad (\text{III-10a})$$

$$\frac{dx_2}{dt} = a_{21}(t) x_1 + a_{22}(t) x_2, \quad (\text{III-10b})$$

the differential equations describing each variable are:

$$\begin{aligned} \frac{d^2 x_2}{dt^2} = & \left(\frac{\dot{a}_{21}}{a_{21}} + a_{11} + a_{22} \right) \frac{dx_2}{dt} + \\ & (-a_{11} a_{22} + a_{12} a_{21} + \dot{a}_{22} - a_{22} \frac{\dot{a}_{21}}{a_{21}}) x_2, \end{aligned} \quad (\text{III-11a})$$

$$\begin{aligned} \frac{d^2 x_1}{dt^2} = & \left(\frac{\dot{a}_{12}}{a_{12}} + a_{11} + a_{22} \right) \frac{dx_1}{dt} + \\ & (-a_{11} a_{22} + a_{12} a_{21} + \dot{a}_{11} - a_{11} \frac{\dot{a}_{12}}{a_{12}}) x_1. \end{aligned} \quad (\text{III-11b})$$

Thus, the differential equations describing $x_1(t)$ and $x_2(t)$ are different, and since the solutions of each of these equations are unique, the solutions will be different. The solutions depend upon the rate of change of

the coefficients with time, as well as their instantaneous values. As the coefficients become constant, the equations become identical.

Denoting III-11b as

$$\frac{d^2 x_1}{dt^2} + A \frac{dx_1}{dt} + B x_1 = 0, \quad (\text{III-12})$$

we may determine the differential equation for $\frac{dx_1}{dt}$ by differentiation and substitution as:

$$\frac{d^2}{dt^2} \left(\frac{dx_1}{dt} \right) + \left[A - \frac{\dot{B}}{B} \right] \frac{d}{dt} \left(\frac{dx_1}{dt} \right) + \left[B + \dot{A} - A \frac{\dot{B}}{B} \right] \left(\frac{dx_1}{dt} \right) = 0. \quad (\text{III-13})$$

Therefore, in general, $\frac{dx_1}{dt}$ will involve different functions of time than x_1 .

Thus, in the study of the dynamics of systems described by time variable equations, we must investigate the nature of all variables. The conventional concepts of stability as applied to constant coefficient systems must be applied with care. In a constant coefficient system, where the stability of one variable ensures the stability of all variables and derivatives of the variables, the characteristic equation of the constant coefficient system indicates directly the form of all variables and their derivatives. This is no longer true in a time varying system.

In a specific problem it is probably not necessary to ensure the stability of all variables, but only certain ones of concern; say in a piloted airplane, those that are sensed by the pilot or that may cause structural damage if they exceed certain limits.

B. Some Well-Known Results

We refer to those solutions readily available in analytic or tabular form. Many functions have been investigated specifically because they are solutions to particular linear differential equations; Bessel functions, Mathieu functions, the hypergeometric function, to name a few. We briefly investigate the nature of some solutions that are well known and readily available, at least in tabular form. Generally, such tabular results are available only for second-order equations. For the differential equation

$$\frac{d^2 \theta}{dt^2} + b(t) \theta = 0, \quad (\text{III-4})$$

a list of tables available for various simple forms of $b(t)$ may be found in Reference 29.

Let us consider $b(t) = t^N$, since we are primarily interested in monotonically varying coefficients. Equation III-4 becomes

$$\frac{d^2 \theta}{dt^2} + t^N \theta = 0. \quad (\text{III-14})$$

Let us determine the nature of the solutions to equation III-14 for various values of N .

$N = 0$

We have a constant coefficient equation,

$$\theta = C_1 \cos(t + \Phi). \quad (\text{III-15})$$

$N = -2$

We have Euler's equation¹⁶ and the solution may be found by assuming a solution of the form $C t^N$,

$$\theta = C_1 t^{\frac{1}{2} + \frac{\sqrt{3}}{2}i} + C_2 t^{\frac{1}{2} - \frac{\sqrt{3}}{2}i} \quad (\text{III-16a})$$

or

$$\theta = t^{\frac{1}{2}} C_1 \cos\left(\frac{\sqrt{3}}{2} \ln t + \Phi\right). \quad (\text{III-16b})$$

$N = -4$

We have an exact solution,

$$\theta = t \left\{ C_2 \cos \left[\left(\frac{1}{t} - \frac{1}{t_0} \right) + \Phi \right] \right\}. \quad (\text{III-17})$$

We note from III-16a and III-17 that the argument of the oscillating term is of a form that might be expected, i. e., $\int_0^t b^{\frac{1}{2}} ds$; however, in addition, we obtain an amplitude modification. Note that in both these examples the amplitude of the response grows without limit as $t \rightarrow \infty$ even though

there is no velocity dependent term in the original equation. This is the sort of time varying effect that we wish to consider in more detail.

N any value

The solution to equation III-14 may be expressed in terms of Bessel functions for any power of N (Reference 18, page 147).

$$\theta = \sqrt{t} \{ C_1 J_p(2p t^{\frac{1}{2p}}) + C_2 N_p(2p t^{\frac{1}{2p}}) \}, \quad (\text{III-18})$$

where J_p is a Bessel function of the first kind and N_p is a Bessel function of the second kind, and $p = \frac{1}{N+2}$.

When $p = \frac{2m+1}{2}$, where m is an integer, J_p and N_p are expressible in a finite number of terms (Reference 17, page 69).

If $N = 1$, equation III-14 is sometimes called Airy's equation (Reference 19). Also in Reference 19, for $-\frac{3}{4} \leq N \leq 2$, the solutions are referred to as generalized Airy functions. Airy functions are Bessel functions with a modified amplitude and argument. Two linearly independent solutions of equation III-14 for $N = 1$ are shown in Figure 1. Both the displacement (θ) and the velocity ($\frac{d\theta}{dt}$) are shown. Note that the displacement amplitude is decreasing with time, and the velocity amplitude is increasing with time. The amplitude of the velocity response is influenced by the changing frequency with time. The difference in the amplitude variation with time of these two quantities points out one property of time variable equations which was discussed earlier.

For $N > 0$, the solutions of III-14 will be qualitatively similar to Figure 1. For $N < 0$, the displacement amplitude will increase and the frequency will decrease, as seen by inspection of equations III-16b and III-17.

Since Bessel functions appear in many engineering problems it is worthwhile to discuss qualitatively their general nature.

The behavior of Bessel functions of the first and second kind for various orders and arguments can be readily visualized from relief maps (Reference 18, pages 152 and 198).

For a sufficiently large value of the argument, Bessel functions are approximately lightly damped oscillations with constant frequency. When the argument is greater than the value corresponding to the first zero,

this is a reasonably accurate description of the function. As the order of the Bessel function is increased, the argument at which the first zero occurs is increased. Simple expressions are available in terms of asymptotic expansions (Reference 18, page 138). As $t \rightarrow \infty$,

$$J_p(t) \rightarrow \frac{\cos \left\{ t - \frac{\pi}{2} \left(p + \frac{1}{2} \right) \right\}}{\sqrt{\frac{\pi t}{2}}} \quad (\text{III-19})$$

Note the appearance of the order p only as a phase shift.

When p is an integer, we must express the two linearly independent solutions, as above, in terms of Bessel functions of the first and second kind. When p is not an integer, we may either use this form or $J_{-p}(x)$ as the other solution.

In some problems it may be convenient to use J_p and N_p , since asymptotically J_p and N_p are 90° out of phase, while J_p and J_{-p} are not.

Note that the behavior of the solutions to equation III-14, for small arguments, is considerably modified by the presence of the additional factor \sqrt{t} , which now makes the term $\sqrt{t} N_p(2p t^{\frac{1}{2p}})$ approach a constant as $t \rightarrow 0$ even though $N(t) \rightarrow \infty$ as $t \rightarrow 0$. Figure 1 shows this behavior clearly.

Bessel functions are solutions to a class of differential equations that represent a wide variety of physical problems, and we see that they enter into the problem considered here as they have into other related problems (References 4, 9, and 10).

Note that for the type of equation under consideration, the argument of the Bessel function is not linearly proportional to the independent variable.

C. Approximate Solutions

We have discussed some well-known solutions and some general properties of linear time varying differential equations. Since these solutions are generally available in tabular and not analytical form and applicable to specific coefficient variations with time, approximate results are of considerable assistance in providing insight into the nature of the solutions to time varying equations for engineering applications.

Approximate solutions to differential equations can be considered from two points of view. We will take the one where the differential equation is examined and approximate solutions are developed for specified restrictions on the coefficients and their variations with time. The alternate point of view, where we consider approximations to the coefficients of the differential equation such that the solution of the differential equation with approximate coefficients may be expressed in analytical form, or at least is available in tabular form, will not be discussed (References 11 and 29). These points of view are equivalent, since the approximate solution obtained from the former approach will exactly satisfy a differential equation with coefficients that are slightly different from the actual differential equation investigated.

We will discuss some approximate solutions to a second-order linear differential equation, as well as higher-order equations, and evaluate the conditions under which various approximate solutions are valid.

1. The Frozen Solution (Reference 27). This solution is obtained from a constant coefficient differential equation based on the initial values of the coefficients of a variable coefficient differential equation. Specifically, for equation III-4,

$$\frac{d^2\theta}{dt^2} + b(t) \theta = 0, \quad (\text{III-4})$$

over a time interval (t_1, t_2) the frozen solution is defined as the solution of:

$$\frac{d^2\theta}{dt^2} + b(t_1) \theta = 0. \quad (\text{III-20})$$

This approximation would be expected to be valid when a solution is desired over a time interval in which the fractional change of the restoring force is small. A more exact estimate of the error is considered later.

In multiple-degree-of-freedom systems, different frozen solutions will result, depending upon whether the frozen system approximation is made directly with the coupled differential equations, or with differential equations describing each variable separately. Given a system described by coupled differential equations, the differential equations describing each variable will contain additional terms, depending upon the rate of change of the coefficients. These new terms arise in the process of eliminating variables.

For example, for a system described by the two coupled differential equations III-10, the frozen system approximation is:

$$\frac{dx_1}{dt} = a_{11}(t_1) x_1 + a_{12}(t_1) x_2, \quad (\text{III-21a})$$

$$\frac{dx_2}{dt} = a_{21}(t_1) x_1 + a_{22}(t_1) x_2, \quad (\text{III-21b})$$

and therefore the frozen differential equations describing each variable are:

$$\begin{aligned} \frac{d^2 x_1}{dt^2} &= (a_{11}(t_1) + a_{22}(t_1)) \frac{dx_1}{dt} + \\ &(-a_{11}(t_1) a_{22}(t_1) + a_{12}(t_1) a_{21}(t_1)) x_1, \end{aligned} \quad (\text{III-22a})$$

$$\begin{aligned} \frac{d^2 x_2}{dt^2} &= (a_{11}(t_1) + a_{22}(t_1)) \frac{dx_2}{dt} + \\ &(-a_{11}(t_1) a_{22}(t_1) + a_{12}(t_1) a_{21}(t_1)) x_2, \end{aligned} \quad (\text{III-22b})$$

However, if, instead, we first eliminate variables and uncouple the differential equations III-10 to obtain III-11, and then make the frozen approximation, we obtain:

$$\begin{aligned} \frac{d^2 x_1}{dt^2} &= (a_{11}(t_1) + a_{22}(t_1) + \frac{\dot{a}_{12}(t_1)}{a_{12}(t_1)}) \frac{dx_1}{dt} + \\ &(-a_{11}(t_1) a_{22}(t_1) + a_{12}(t_1) a_{21}(t_1) + \dot{a}_{11}(t_1) - a_{11}(t_1) \frac{\dot{a}_{12}(t_1)}{a_{12}(t_1)}) x_1, \end{aligned} \quad (\text{III-23a})$$

$$\begin{aligned} \frac{d^2 x_2}{dt^2} &= (a_{11}(t_1) + a_{22}(t_1) + \frac{\dot{a}_{21}(t_1)}{a_{21}(t_1)}) \frac{dx_2}{dt} + \\ &(-a_{11}(t_1) a_{22}(t_1) + a_{12}(t_1) a_{21}(t_1) + \dot{a}_{11}(t_1) - a_{22}(t_1) \frac{\dot{a}_{21}(t_1)}{a_{21}(t_1)}) x_2, \end{aligned} \quad (\text{III-23b})$$

where, in general, $\dot{a}_{12}(t_1)$, $\dot{a}_{21}(t_1)$, $\dot{a}_{11}(t_1)$, and $\dot{a}_{22}(t_1)$ are not zero. Thus equations III-22 differ from equations III-23.

For our purposes, it is convenient to define the frozen solution as that resulting from the former method. Therefore, the frozen approximation to equation III-10 is obtained from equation III-22. No terms dependent upon the rate of change of the coefficients will be present in the frozen approximation as defined.

By defining the frozen approximation on this basis, in the case of aircraft dynamics, the frozen approximation will be closely related to the dynamics computed from the classical approach in steady flight. They will differ only due to the fact that in unsteady flight an airplane may encounter flight conditions different from steady flight (Appendices I and III). Thus, the frozen approximation provides a useful reference point from which to evaluate the effects of acceleration and deceleration of the aircraft.

2. The Quasi-Steady Solution²⁸. It might be expected that the range of validity of the frozen solution could be extended by allowing the characteristic roots computed by using the frozen approximation to vary with time. That is, the solution of a variable coefficient equation is approximated in the following way. A characteristic equation is determined as though the differential equation had constant coefficients. The roots of this fictitious characteristic equation are computed as a function of time. The solution is assumed to be

$$x = c_1 e^{\int_0^t r_1(s) ds} + c_2 e^{\int_0^t r_2(s) ds}, \quad (\text{III-24})$$

where $r_1(t)$ and $r_2(t)$ are the roots obtained from the fictitious characteristic equation at various times. This approximate solution will be defined as the quasi-steady solution.

Therefore, the quasi-steady solution to the equation III-4 is:

$$\theta = c_1 \cos \left\{ \int_0^t b^{\frac{1}{2}} ds + \Phi_1 \right\}. \quad (\text{III-25})$$

The rationale for this approximate solution is based on the solution of a first order differential equation; e. g. ,

$$\frac{d\theta}{dt} + a(t) \theta = 0, \quad (\text{III-26})$$

which has the exact solution

$$\theta = \theta_0 e^{-\int_0^t a(s) ds}. \quad (\text{III-27})$$

Thus, for a first-order equation, the quasi-steady solution is the exact solution. Unfortunately, it is no longer exact for higher-order equations.

Again, the quasi-steady solution will be different, depending upon how "roots" of the system are computed, using the coupled differential equations or the uncoupled differential equations for each variable. We will define the quasi-steady solution as that resulting from the uncoupled differential equations, thus including terms involving the rate of change of the coefficients in the computation of the "roots." The quasi-steady solution to the system of equation III-10 is therefore defined as that obtained by computing the "roots" to the characteristic equations corresponding to III-23 at various instants of time in the time interval of interest and placing these "roots" in the assumed exponential form of the solution. The quasi-steady solution will reflect a difference in the various variables in a multiple-degree-of-freedom system.

With this definition, therefore, in the following we will refer to the effects that appear due to these terms that appear upon decoupling these equations, still considering that the equations are solved as constant coefficient equations, as quasi-steady effects.

The solution of a linear differential equation with variable coefficients depends upon the fact that the coefficients are actually changing with time as well as the appearance of these new terms. Note that no effects on the nature of the solution are taken into account in the quasi-steady approximation. We have only guessed that this form of the solution might be an improvement over the frozen solution, and it remains to be seen whether the quasi-steady solution reflects a real improvement over the frozen solution. This point is considered in the section on error estimates.

We now wish to consider approximate methods of taking into account the influence of variable coefficients on the nature of the

solutions. As might be expected, it appears difficult to find any one approximate solution that is generally applicable; however, there are two approximate solutions that are useful for the problem at hand.

3. The Asymptotic Solution^{16,22} (Unsteady Effects). If the coefficients of the differential equations possess certain properties, it might be expected that there are approximate solutions that apply for various time intervals. In particular, the approximate solution that the exact solution approaches as $t \rightarrow \infty$ is called the asymptotic solution^{16,22}.

For a second-order equation, application of the transformations III-2 and III-5 results in a differential equation in which the varying nature of the coefficients appears added to a constant. Transforming equation III-4, first using transformation III-5,

$$t_1 = \int_0^t b^{\frac{1}{4}} ds, \quad (\text{III-5})$$

and then using III-2 to remove the damping term,

$$\theta_1 = \theta e^{\frac{1}{4} \int_0^t \frac{1}{b} \frac{db}{ds} ds} = \theta b^{\frac{1}{4}}, \quad (\text{III-2})$$

the resulting differential equation is:

$$\frac{d^2 \theta_1}{dt_1^2} + \left\{ 1 - \frac{1}{b^{\frac{1}{4}}} \frac{d^2}{dt_1^2} (b^{\frac{1}{4}}) \right\} \theta_1 = 0 \quad (\text{III-28})$$

If

$$\frac{1}{b^{\frac{1}{4}}} \frac{d^2}{dt_1^2} (b^{\frac{1}{4}}) = \frac{1}{b^{\frac{3}{4}}} \frac{d^2}{dt_1^2} (b^{\frac{1}{4}}) \rightarrow 0, \quad (\text{III-29})$$

as $t_1 \rightarrow 0$, equation III-28 approaches,

$$\frac{d^2 \theta_1}{dt_1^2} + \theta_1 = 0. \quad (\text{III-30})$$

The general solution of III-30 is:

$$\theta_1 = C_1 \cos (t_1 + \Phi). \quad (\text{III-31})$$

If expression III-29 has suitable properties, then III-31 is the asymptotic solution to III-4. In terms of the untransformed variables, the approximate solution is:

$$\theta = \frac{1}{b^{\frac{1}{4}}} \left\{ C_1 \cos \left[\int_0^t b^{\frac{1}{2}} ds + \Phi_1 \right] \right\}. \quad (\text{III-32})$$

It has been shown in Reference 22 that III-32 is the asymptotic solution to III-4 when

$$\int_{t_0}^{\infty} \frac{1}{b^{\frac{1}{4}}} \left| \frac{d^2}{dt^2} (b^{-\frac{1}{4}}) \right| dt < \infty \quad b(t) > 0. \quad (\text{III-33})$$

This approximate solution, III-32, to equation III-4 is known as the WKBJ method (References 12 and 29), or in some cases as the Liouville-Green method (Reference 30).

In many specific cases, the criterion III-32 will hold. For example, if we obtain an approximate solution to Bessel's equation in this way, the asymptotic expansion of Bessel functions will appear as solutions. Equation III-32 is the exact solution to III-4 when $b \propto t^{-4}$, and results in a constant error in frequency when applied to Euler's equation.

Comparison of the asymptotic solution of III-4, III-32, with the quasi-steady solution III-25, shows that the asymptotic solution leads to the same frequency term as the quasi-steady solution, and in addition yields an amplitude variation, or apparent damping, due to the varying spring constant.

4. Initial Value Approximation. It might be expected that a simple approximation would apply when the time interval of interest is small compared to the average period of the motion.

If we consider the transformed differential equation III-28,

$$\frac{d^2\theta_1}{dt_1^2} + \left\{ 1 - \frac{1}{b^{\frac{1}{4}}} \frac{d^2}{dt_1^2} (b^{\frac{1}{4}}) \right\} \theta_1 = 0, \quad (\text{III-28})$$

to obtain the asymptotic solution, we neglect the second term in the restoring force by comparison to 1. Now consider the other extreme, where this term dominates the restoring force, such that we approximate equation III-28 by:

$$\frac{d^2\theta_1}{dt_1^2} - \frac{1}{b^{\frac{1}{4}}} \frac{d^2}{dt_1^2} (b^{\frac{1}{4}}) \theta_1 \cong 0. \quad (\text{III-34})$$

The exact solution of equation III-34 is:

$$\theta_1 = \left\{ C_0 + C_1 \int_0^t b^{-\frac{1}{2}} ds \right\} b^{\frac{1}{4}} \quad (\text{III-35})$$

In terms of the untransformed variable t , the solution is:

$$\theta = C_0 + C_1 t. \quad (\text{III-36})$$

Thus, this limiting case is equivalent to solving equation III-4 by neglecting the restoring force term.

D. Error Estimates for Approximate Solutions

To apply the foregoing approximations, it is desirable to have some estimate of their accuracy or criteria for our use. Here we will consider the error incurred in using the various approximations.

1. Frozen and Quasi-Steady Approximations. Given the differential equation

$$\frac{d^2\theta}{dt^2} + b(t) \theta = 0, \quad (\text{III-4})$$

it should be satisfactory to expand $b(t)$ about its initial value using a Taylor series and retain only the first-order term, since it is physically clear that the frozen approximation applies only for small coefficient variations. Then:

$$\frac{d^2\theta}{dt^2} + (b_0 + \dot{b} t) \theta = 0. \quad (\text{III-37})$$

Nondimensionalize the time by $\tau = \sqrt{b_0} t$, and let

$$\epsilon = \frac{1}{b_0} \frac{db}{d\tau}. \quad (\text{III-38})$$

$(2\pi \epsilon)$ is the fractional change of the coefficient b in one initial period. Then:

$$\frac{d^2\theta}{d\tau^2} + (1 + \epsilon \tau) \theta = 0. \quad (\text{III-39})$$

The differential equation III-39 may be converted into an integral equation by regarding $\epsilon \tau \theta$ as a forcing term:

$$\theta = C_1 \sin \tau + C_2 \cos \tau + \epsilon \int_0^{\tau} \sin(s - \tau) s \theta(s) ds. \quad (\text{III-40})$$

The zeroth approximation to III-40 for small ϵ is obtained by neglecting the integral term to obtain

$$\theta_0 = C_1 \sin \tau + C_2 \cos \tau. \quad (\text{III-41})$$

This is the frozen approximation. Now, we iterate, to obtain a first approximation by placing θ_0 under the integral sign. Evaluating the integral results in the following expression for θ_1

$$\begin{aligned} \theta_1 = & \left[1 - \frac{\epsilon \tau}{4} \right] [C_1 \sin \tau + C_2 \cos \tau] + \\ & \frac{\epsilon \tau^2}{4} [C_1 \cos \tau - C_2 \sin \tau] + \frac{C_2 \epsilon}{4} \sin \tau. \end{aligned} \quad (\text{III-42})$$

The last term is a constant amplitude term, reflecting no change in the nature of the solution. The middle terms represent

a frequency change and are second order. The first-order effect is shown by the amplitude modification to the frozen solution (III-41), due to the term $\frac{\epsilon \tau}{4}$, where $\epsilon \tau$ is the total fractional coefficient change over the time of observation. It is important to note that the first-order effect is on the amplitude of the motion and not on the frequency.

The quasi-steady approximation, i. e., taking into account the frequency change with time, would represent a partial improvement over the frozen approximation. When the frequency variation is large enough to be important, however, there would also be a significant amplitude change.

If it is desired to limit the amplitude error when using the frozen approximation to say 5 percent, then the total coefficient change is limited to 20 percent. This approximation, as anticipated, is valid for a given total fractional coefficient change.

2. Asymptotic Solution. Error bounds for the asymptotic solution have been developed in Reference 30. We restate only the result that applies to the oscillatory case (theorem 4). $b(t) > 0$ and $\frac{d^2 b}{dt^2}$ is continuous over the interval $a \leq t \leq b$. Then the differential equation

$$\frac{d^2 \theta}{dt^2} + b(t) \theta = 0 \quad (\text{III-4})$$

has conjugate solutions, θ, θ^* , where

$$\theta = \frac{1}{b^{\frac{1}{4}}} \left[e^{i \int_0^t b^{\frac{1}{2}} ds} + \epsilon \right] \quad (\text{III-43})$$

where

$$\epsilon \leq e^{F(t)}, \quad (\text{III-44a})$$

and

$$F(t) = \int_c^t \frac{1}{b^{\frac{1}{4}}} \left| \frac{d^2}{dt^2} \left(b^{-\frac{1}{4}} \right) \right| dt; \quad (\text{III-44b})$$

c is an arbitrary point in the interval; i. e., the time at which the initial conditions are applied (the solution is then exact at this point). Uniform bounds on the error may be obtained by evaluating the integral (III-44b) over the entire interval. If we let the interval become infinite; i. e., $c \leq t \leq \infty$, then we note that the condition for F to be finite is the condition that the approximate solution (III-32) is the asymptotic solution.

If $\frac{d^2}{dt^2} \left(b^{-\frac{1}{4}} \right)$ is of uniform sign, III-44b may be integrated by parts to give

$$F(t) = \frac{1}{b^{\frac{1}{4}}} \frac{d}{dt} \left(b^{-\frac{1}{4}} \right) \Bigg|_c^t - \int_0^t \left\{ \frac{d}{dt} \left(b^{-\frac{1}{4}} \right) \right\}^2 dt \quad (\text{III-45a})$$

or

$$F(t) = \frac{1}{4} \frac{\dot{b}}{b^{\frac{3}{2}}} \Bigg|_c^t - \int_0^t \frac{1}{b^{\frac{3}{2}}} \left\{ \frac{1}{4} \frac{\dot{b}}{b^{\frac{3}{2}}} \right\}^2 dt. \quad (\text{III-45b})$$

From III-45b we see that the parameter that determines the error and thus the applicability of the asymptotic solution is $\frac{\dot{b}}{b^{\frac{3}{2}}}$, the instantaneous fractional change in the restoring force per cycle.

When this parameter is sufficiently small, we may use the approximate solution III-32 and estimate the error incurred by use of equations III-44.

This approximate solution is only dependent upon the rate of change of the coefficient, and not on the total coefficient change, thus representing a definite improvement over the frozen and quasi-steady approximation.

The two contributions to the error, as indicated by equation III-45, may be interpreted in the following way. The integral term represents a frequency error due to the neglect of the term

$\frac{1}{b^{\frac{1}{4}}} \frac{d^2}{dt^2} \left(\frac{1}{b^{\frac{1}{4}}} \right)$ in the restoring force term of equation III-28 when

computing the frequency. This error will increase with time, even if $\frac{b}{3}$ is constant, since it is a frequency error, thus ac-

counting for the appearance of the integral. The increasing magnitude of this error with time is not of particular concern, since we are primarily concerned with the nature of the solution and not with the exact coordinates of the vehicle at any time.

The first term corresponds to an error in the amplitude of the response, since it is a measure of the variation in the restoring force term of the transformed equation, III-28. If the two transformations, III-2 and III-5, were applied to equation III-28, it can be seen that the variation in $\frac{b}{3}$ would be reflected as an ad-

ditional amplitude change.

In specific problems, it may be possible to refine the solutions by repeating the transformations before neglecting any terms in the coefficients.

Let us see what these results imply regarding the application of the asymptotic solution to the specific instance where $b \propto t^N$.

The condition that the approximate solution be the asymptotic solution is from III-33:

$$\int_c^\infty \left| \frac{1}{N} \frac{d^2}{dt^2} \left(t^{\frac{N}{4}} \right) \right| dt = \frac{5}{16} N \int_c^\infty \frac{dt}{t^{\frac{N}{2}+2}} = \frac{5}{16} N \left\{ \frac{t^{1+\frac{N}{2}}}{1+\frac{N}{2}} \right\}_c^\infty < \infty \quad \{ N \neq -2 \}$$

(III-46)

Thus, whenever $N > -2$, the approximate solution is the asymptotic solution. When $N = -2$, we have Euler's equation. When $N < -2$, then the solution is no longer the asymptotic solution. However, since this term is also an estimate of the error

we may still be able to use the approximate solution when $N < -2$, but now only for short times, since the error is increasing with time.

These three cases correspond to a different behavior of the fractional change in the coefficient with time. The fractional change in the coefficient per cycle is

$$\frac{\dot{b}}{b} = \frac{N}{t(1 + \frac{N}{2})} \quad (\text{III-47})$$

When $N > -2$, then the fractional change per cycle decreases; when $N = -2$, the fractional change per cycle is constant (the differential equation is Euler's equation), and the original equation can be solved exactly. When $N < -2$, the fractional change per cycle increases; the approximate solution is no longer the asymptotic solution, but may apply over some range of time, depending upon the exact nature of the coefficient variation.

We would therefore expect that the expression, III-32, approximates the solution to equation III-4 quite well when the magnitude of the transformed spring constant is increasing, and has limitations when the magnitude of the spring constant is decreasing.

For a system with an increasing spring constant, the asymptotic solution is a long-time solution, and for a system with a decreasing spring constant it is a short-time solution.

To obtain a numerical indication of the size of the error, we investigate a simple example where the spring constant varies linearly with time.

Let $b = 1 + \epsilon t$, and evaluate the error for $0 < t < \infty$. Therefore,

$$\frac{\dot{b}}{b} = \frac{\epsilon}{(1 + \epsilon t)} \quad \text{and} \quad F(\infty, c) = \int_c^\infty \frac{1}{b^{\frac{1}{4}}} \left| \frac{d^2}{dt^2} \left(b^{-\frac{1}{4}} \right) \right| dt = \frac{5}{24} \epsilon \left[\frac{1}{(1 + \epsilon t)^{\frac{3}{2}}} \right]_c^\infty \quad (\text{III-44b})$$

when $c = 0$:

$$F(\infty) = \frac{5}{24} \epsilon.$$

Thus the error is proportional to ϵ , and if we require the error to be less than 5 percent, then $\epsilon \cong .25$. From foregoing considerations, it is difficult to interpret exactly what this 5 percent error represents in the way of differences between the exact and approximate solutions. This variation in b represents the coefficient increasing by a factor of $2\frac{1}{2}$ in a cycle.

When $b(t)$ is decreasing with time ($\epsilon < 0$) the error will increase with time, and so the criterion will give a final value of time up to which the asymptotic approximation applies, since $1 + \epsilon \tau$ is becoming smaller.

Another interesting ramification of this approximate solution may be seen from the previous discussion. Consider the second-order differential equation in terms of a damper and a spring constant:

$$\frac{d^2x}{dt^2} + c(t) \frac{dx}{dt} + k(t) x = 0, \quad (\text{III-1})$$

and the transformed equation III-8:

$$\frac{d^2\theta}{ds^2} + \left(1 - (\zeta')^2 - \frac{d\zeta'}{ds} \right) \theta = 0, \quad (\text{III-8})$$

where

$$\zeta' = \left[\frac{c(s)}{2\sqrt{k(s)}} + \frac{1}{4} \frac{1}{k(s)} \frac{dk}{ds} \right]. \quad (\text{III-7})$$

Thus, it appears that we only require that in physical terms the resultant damping of the system be small to use the asymptotic solution; i. e., if the variable spring has a significant contribution to the damping of the system, it still may be possible to use the approximate solution when $c(t)$ is of such a magnitude to make the term ζ' and its derivative small. In other words, this implies that the asymptotic solution applies whenever the damping of the system is small.

3. The Initial Value Approximation. This approximation is the solution obtained when the spring constant is neglected and is the limiting case of very large percentage spring constant variation.

To estimate the error involved, we convert the differential equation

$$\frac{d^2 \theta}{dt^2} + b(t) \theta = 0 \quad (\text{III-4})$$

into an integral equation:

$$\theta = C_1 + C_2 t + \int_0^t (s - t) b(s) \theta(s) ds, \quad (\text{III-48})$$

the integral term being a measure of the error, since it is neglected in the approximate solution III-36.

To estimate the size of this term substitute $\theta_0 = C_1 + C_2 t$ under the integral sign. For simplicity, assume that b may be replaced by its average value \tilde{b} and obtain

$$\theta(t) = (C_1 + C_2 t) \left(1 - \frac{1}{2} \tilde{b} t^2\right). \quad (\text{III-49})$$

Thus $\tilde{b} t^2$ must be small compared to 1.

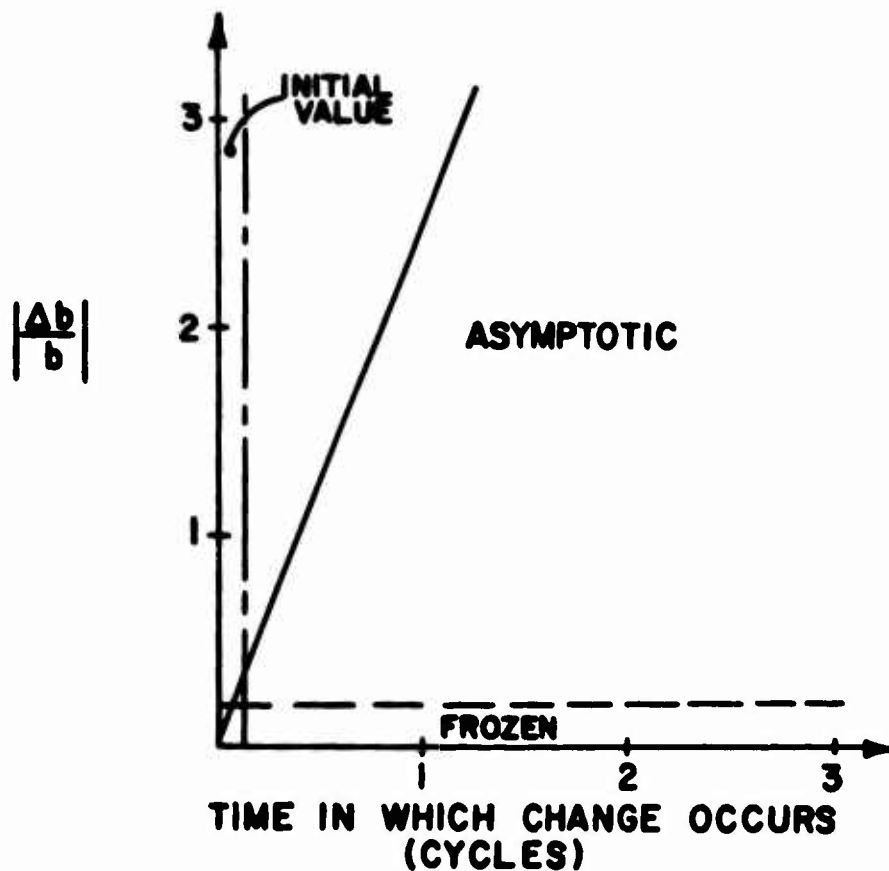
$$\tilde{b} t^2 = 4\pi^2 \left(\frac{t}{p}\right)^2. \quad (\text{III-50})$$

If the time interval of interest is less than one-eighth of a period, the error incurred in this approximation is small. This approximation is particularly useful when the restoring force is decreasing.

If we wish to limit the error to 5 percent then we are limited to about 5 percent of the average period in time. This approximation is directly dependent upon the time of observation, compared to the average frequency of motion.

E. Relationships Between Approximate Solutions

The regions of applicability of the various approximations to the differential equation III-4 may be displayed graphically in the following fashion.



For a given system we move around on this graph, depending on the variation of restoring force term. Thus, various approximations may apply over different intervals of time.

How the quasi-steady approximation fits into this picture is more difficult to estimate. Its validity is specifically tied to the nature of the coefficients in the differential equation under consideration.

Comparison of the asymptotic approximation and the quasi-steady approximation indicates that the quasi-steady approximation does not reflect the apparent damping caused by a variable restoring force term, while the asymptotic approximation does. This indicates that, generally, for second-order equations, the quasi-steady approximation would be a reasonable approximation to a system with slowly varying coefficients, if the system had a reasonably high level of inherent damping.

The quasi-steady approximation and the asymptotic approximation will be the same for second-order systems with a constant restoring force. This may be seen by applying the transformation, III-5, to equation III-4:

$$\frac{d^2\theta}{dt_1^2} + \frac{1}{2} \frac{1}{b} \frac{db}{dt_1} \frac{d\theta}{dt_1} + \theta = 0. \quad (\text{III-51})$$

If we solve this equation using the quasi-steady approximation, neglecting the effects of the damping term on the frequency, we obtain:

$$\theta = c_1 e^{-\frac{1}{4} \int_0^{t_1} \frac{1}{b} \frac{db}{ds} ds} \cos(t_1 + \Phi_1). \quad (\text{III-52a})$$

The exponent is an exact differential, and thus we have the asymptotic approximation:

$$\theta = c_1 \frac{1}{b^{\frac{1}{4}}} \cos\left(\int_0^{t_1} \frac{1}{b^{\frac{1}{2}}} ds + \Phi_1\right). \quad (\text{III-52b})$$

Thus the quasi-steady approximation, when applied to an equation with a constant restoring force and variable damping, gives the same result as the asymptotic approximation. However, different results are obtained when the system has a variable spring force (cf. equations III-32 and III-25 as the solution of III-4).

The degree with which the quasi-steady approximation represents the first-order effects of time variation of the coefficients depends to a large extent on the nature of the time variable coefficients. Let us consider the various possibilities with respect to the system of differential equations, III-10. We concentrate our attention on the restoring force terms in equations III-11.

If the restoring force term can be approximated by $a_{11} a_{22}$, i. e., if the coupling is weak and if $a_{11} a_{22}$ is constant, then we would expect that the quasi-steady solution represents a good approximation, since the restoring force in each equation III-11 is approximately constant. If, however, we have the opposite situation, largely typical of the dynamic stability equations of aircraft, where coupling terms dominate the restoring

force such that the restoring force may be approximated by $a_{12} a_{21}$, then the apparent effect due to the presence of the new terms in the damping coefficient may be entirely spurious because of the influence of the varying spring on the amplitude of the motion. This effect becomes apparent by transforming the differential equations III-11 such that the spring force is a constant, and then applying the quasi-steady approximation. Applying the independent variable transformation, equation III-5, assuming that the restoring force is approximately $(- a_{12} a_{21})$, to equations III-11,

$$t_1 = \int^t \sqrt{-a_{12} a_{21}} \, ds, \quad (\text{III-53})$$

results in the differential equations

$$\frac{d^2 x_1}{dt^2} + \left[-a_{11} - a_{22} + \frac{1}{2} \left(\frac{\dot{a}_{21}}{a_{21}} - \frac{\dot{a}_{12}}{a_{12}} \right) \right] \frac{1}{\sqrt{-a_{12} a_{21}}} \frac{dx_1}{dt} + x_1 = 0, \quad (\text{III-54a})$$

$$\frac{d^2 x_2}{dt^2} + \left[-a_{11} - a_{22} + \frac{1}{2} \left(\frac{\dot{a}_{12}}{a_{12}} - \frac{\dot{a}_{21}}{a_{21}} \right) \right] \frac{1}{\sqrt{-a_{12} a_{21}}} \frac{dx_2}{dt} + x_2 = 0. \quad (\text{III-54b})$$

The quasi-steady solution of equations III-54 will indicate quite different effects on the amplitude of the solution than the quasi-steady solution of equations III-11, particularly when the inherent damping of the system $(a_{11} + a_{22})$ is small, or when the system is near the boundary of stability.

Therefore, one must be very careful in estimating the first-order effects of time varying coefficients solely on the basis of the appearance of new terms in the differential equations. These terms do give rise to differences in the functions of time describing the various variables, but in general will only reflect part of the first-order effects of varying coefficients.

To summarize, the frozen and quasi-steady approximations do not take into account any effects that the changing coefficients may have on the nature of the solutions to the differential equation. Essentially, the differential equations are solved as through they were constant coefficient equations. In the frozen system, all the coefficients remain fixed at their initial values.

The quasi-steady solution includes changes in the appearance of the differential equations due to coefficient variations; but, when solving the differential equations, considers the coefficients stationary in time, and may or may not represent an actual improvement over the frozen approximation.

With regard to coupled systems, the foregoing investigation indicates the following trends with respect to the relative importance of quasi-steady and unsteady effects. The quasi-steady effects are those dependent upon the appearance of new terms obtained in the process of uncoupling differential equations describing the motion, and the unsteady effects are those direct influences on the nature of the solutions of the differential equations due to the time varying coefficients. For purposes of this discussion we consider unsteady effects as those indicated by the asymptotic solution. In the second-order case the primary effect is that of the variable restoring force on the apparent damping of the motion.

For example, if the coupling terms (a_{12} a_{21}) in equations III-11 are varying, and also dominate the frequency of the motion, then the quasi-steady effects and the unsteady effects will be of a similar magnitude.

If the coupling terms (a_{12} a_{21}) are varying, but the uncoupled terms (a_{11} a_{22}) dominate the frequency of the motion, then the relative size of the two effects would depend upon the relative size of the fractional rates of change of the coupling and uncoupled terms.

If the coupling terms are constant, then there will be no quasi-steady effects in this case, but there may be unsteady effects from variation in the uncoupled terms (a_{11} a_{22}) with time, assuming that they dominate the frequency of the motion.

Thus we cannot, in general, decide upon the validity of the quasi-steady or frozen approximation on the basis of the quasi-steady terms, as suggested in References 12 and 21. Note that the short period equations of an aircraft fall in the category in which the frequency of the motion is determined by coupling terms (Appendix III, equations B-36 and B-37).

$$\frac{dw}{dt} = Z_w w + U q, \quad (B-36)$$

$$\frac{dq}{dt} = M_w w + M_q q. \quad (B-37)$$

The short period frequency generally is determined by the coupling terms M_w and U .

A physical interpretation of the two variable transformations used is interesting. Consider equation III-4:

$$\frac{d^2\theta}{dt^2} + b(t)\theta = 0. \quad (\text{III-4})$$

Transforming the independent variable by

$$t_1 = \int_0^t \sqrt{b} ds, \quad (\text{III-5})$$

the equation becomes a differential equation with a variable damping term and a constant restoring force.

$$\frac{d^2\theta}{dt_1^2} + 2\zeta(t_1) \frac{d\theta}{dt_1} + \theta = 0,$$

where $\zeta(t_1) = \frac{1}{b(t_1)} \frac{db}{dt_1}$. Thus, we may consider that the changing spring constant gives rise to damping. An increasing spring constant with time gives rise to a reduction in the displacement amplitude, and a decreasing spring constant gives rise to an increasing displacement amplitude. When we transform the dependent variable so as to remove the damping terms, we obtain

$$\frac{d^2\theta_1}{dt_1^2} + (1 - \zeta^2 - \frac{d\zeta}{dt_1})\theta = 0, \quad (\text{III-55})$$

where we have the familiar effect of damping on frequency (ζ^2) and an additional effect due to the time variation of the damping term ($\frac{d\zeta}{dt_1}$).

Thus, in this sense, the slowly varying coefficient case corresponds to a small ζ , or a small effect on the frequency in the transformed equation, and a very rapid coefficient variation corresponds to a $\zeta \gg 1$ or a heavy damping.

The transformation to remove the damping term indicates a small additional effect of the damping varying with time on the character of the system.

Generally the variable spring effects on the amplitude of the motion would be of most interest and importance in studying physical systems of the type considered here.

We note from these transformations that the importance of the variable damping is shown by comparing the rate of change of the damping ratio with time to unity (see equation III-55). The importance of the variable spring is indicated by comparing the instantaneous values of the fractional change in spring constant per cycle to the instantaneous damping ratio (see equation III-6).

To conclude this chapter, we consider how the foregoing approximations relate to the problem of interest.

The differential equations describing the motion of an aircraft contain coefficients, i. e., stability derivatives, that are functions of velocity. We will assume that the equations of motion may be separated into two parts, one describing the variation of the velocity with time and the other linearized equations describing the perturbation motions. This separation is discussed in detail in Chapter V.

One aspect of the problem is to evaluate how the perturbation response of an aircraft is affected by proceeding from one velocity to another in different time intervals.

As an example, we consider the following simple case.

The differential equation describing the perturbed motion of an aircraft in its simplest form is assumed to be

$$\frac{d^2 \theta}{dt^2} + (1 + \epsilon t) \theta = 0, \quad (\text{III-56})$$

where the parameter ϵ is a function of the acceleration. We are interested in the solution of this differential equation over a time interval $0 \leq \epsilon t \leq a$ which would represent a given percentage change in the flight speed of the airplane. The time is nondimensionalized so that the range of the dependent variable τ is constant.

$$\frac{d^2 \theta}{d\tau^2} + \frac{(1 + \tau)}{\epsilon^2} \theta = 0 \quad (0 \leq \tau \leq a). \quad (\text{III-57})$$

When ϵ is small, we have what could be called a high-frequency problem; there will be a number of cycles present over the time interval of interest,

$$\frac{\dot{b}}{b^{\frac{3}{2}}} \sim \epsilon$$

is small, and the asymptotic solution applies.

If ϵ is large, then for the given range of τ we would be investigating a small time compared to the average period of the motion and the initial value approximation would apply, in what could be termed a low-frequency problem.

Note that when the problem is phrased in these terms, the total coefficient change is fixed and it occurs in a different number of "average" cycles.

When the asymptotic solution applies, III-32, the dependence of the amplitude of the motion upon the change in the spring constant is easily seen. The amplitude of the motion is only dependent upon the instantaneous value of the spring constant. Thus, the apparent damping ratio increases as the time interval for the maneuver becomes shorter if the spring constant is increasing; i. e., we always have the same amplitude change independent of the number of cycles. If the spring constant is decreasing, then we have the opposite effect.

The asymptotic approximation, as we have noted, restricts the effects of a variable spring on the motion to fairly low damping ratios; i. e., we can only apply the asymptotic solution when there is a sizeable coefficient change, if there are a number of cycles present.

F. Summary

For systems described by variable coefficient differential equations, different variables and the derivatives of the variables may involve different functions of time. For example, the amplitude of one variable may be decreasing with time, while the amplitude of another variable may be increasing with time.

The asymptotic solution to second-order linear differential equations is a useful approximation, generally applying to systems where the fractional rate of change of the coefficients with time is small. Physically, this approximate solution appears to apply to systems with light damping. The other approximations discussed in this chapter apply under special circumstances. In particular, care must be taken in applying the quasi-steady approximation, as the only justification for its use is its simplicity.

The primary effect on the response of a second-order system due to variable coefficients is indicated to be an amplitude variation arising from a variation of the restoring force with time. The influence of variable damping on the frequency of a system appears normally to be rather small.

CHAPTER IV

APPROXIMATE SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS OF HIGHER THAN SECOND ORDER WITH VARIABLE COEFFICIENTS

For the application of approximate solutions of linear differential equations with variable coefficients to airplane dynamics, we examine a new interpretation of the asymptotic solution for second-order equations. We then develop new solutions for higher-order systems by a similar approach.

This approach has the advantage of eliminating the need for any variable transformations. We deal directly with the original differential equation in evaluating the nature and magnitude of the unsteady effects.

We commence with the basic idea that, to any linear differential equation of order n there corresponds a nonlinear differential equation of order $n - 1$ (References 16 and 17). From the linear equation, we obtain the nonlinear equation by transforming the dependent variable from x to λ

by relationship $x = e^{\int_0^t \lambda(s) ds}$.

The nonlinear first-order equation that corresponds to a linear second-order equation is called the Ricatti equation (References 16, 17, and 20).

We may interpret this dependent variable transformation as an assumption that the linear equation has a solution of the form

$C e^{\int_0^t \lambda(s) ds}$. The differential equation describing λ may then be considered as the "characteristic equation" of the time-varying equation. We recall our basic premise, that the solutions to the problem of interest will be similar in form to the solutions of constant coefficient equations. A perturbation approach is used to solve this nonlinear equation in order to obtain an approximate solution to higher-order time-varying equations. This is the approach used by Jeffreys (Reference 20) to obtain the WKBJ solution.

A modified root locus technique is developed to interpret these results on the complex plane.

The second-order case is examined from this point of view and then an approximate solution for the n^{th} order equation is developed. Also, some specific remarks are made concerning third-order equations. We will phrase the discussion of the second-order equation in physical terms and then formulate the n^{th} order approximate solution in precise terms.

The Second-Order Equation

First, consider the second-order equation

$$\frac{d^2x}{dt^2} + a(t) \frac{dx}{dt} + b(t) x = 0. \quad (\text{IV-1})$$

Transform the dependent variable by

$$x(t) = C e^{\int_0^t \lambda(s) ds} \quad (\text{IV-2})$$

or

$$\lambda(t) = \frac{1}{x} \frac{dx}{dt}.$$

Rather than calling this a transformation of the dependent variable, we prefer to think of this step as an assumption that the solution of IV-1 takes the form IV-2. We imply that we expect that the solution to equation IV-1 will be similar to the constant coefficient case; i. e., we assume that the solution to IV-1 will be of the form

$$x(t) = C_1 e^{\int_0^t \lambda_1(s) ds} + C_2 e^{\int_0^t \lambda_2(s) ds}. \quad (\text{IV-2a})$$

This assumption is quite reasonable for the problem under consideration where the physical system passes through a series of conditions, in which, at each of these conditions, it is described by a constant coefficient equation. Substituting expression IV-2a into the differential equation IV-1, we obtain two identical nonlinear differential equations of the first order

$$\frac{d\lambda_i}{dt} + \lambda_i^2 + a(t) \lambda_i + b(t) = 0 \quad i = (1, 2) \quad (\text{IV-3})$$

Equation IV-3 is the Ricatti equation as previously mentioned.

By analogy to the constant coefficient case we consider equation IV-3 as an unsteady "characteristic equation" for equation IV-1 and still refer to the λ_i as "roots." We will use this terminology in the following discussion, although neither of these descriptions is precisely true.

It is, of course, no easier to solve the differential equation, IV-3, than to solve the linear equation, IV-1; in fact, in general, one would expect that the linear equation is easier to solve. However, it is easier to develop approximate solutions to the nonlinear equation for the case of interest since λ is assumed to vary slowly by comparison with the modes of motion of the system. We give the procedure in some detail, although the same results were presented in Chapter III, since we shall follow a similar procedure for higher-order systems.

The assumptions will become clear as we proceed with the method. Basically, we assume that the λ_i , i. e., the unsteady "roots," are close to the quasi-steady roots, r_i , such that equation IV-3 may be linearized. We obtain two particular solutions, one associated with each quasi-steady root. These two particular solutions, $\lambda_1(t)$ and $\lambda_2(t)$, may be combined to give the general solution to the Ricatti equation (Reference 17, page 64).

$$\lambda = \frac{C_1 \lambda_1 e^{\int \lambda_1 dt} + C_2 \lambda_2 e^{\int \lambda_2 dt}}{C_1 e^{\int \lambda_1 dt} + C_2 e^{\int \lambda_2 dt}} \quad (IV-4)$$

Comparison of equation IV-4 with the form of the general solution to the second-order linear equation IV-1 and the variable transformation IV-2, indicates that the two particular solutions, λ_1 and λ_2 , may be associated with the two linearly independent solutions of IV-1. The form of the general solution to IV-1 is:

$$x = C_1 U_1(t) + C_2 U_2(t).$$

Substituting into the transformation equation IV-2,

$$\lambda = \frac{1}{x} \frac{dx}{dt} = \frac{C_1 \frac{dU_1}{dt} + C_2 \frac{dU_2}{dt}}{C_1 U_1 + C_2 U_2} \quad (IV-5)$$

Comparison of IV-5 with IV-4 shows that the solutions to IV-1 may be expressed as:

$$U_1 = e^{\int_0^t \lambda_1(s) ds} \quad U_2 = e^{\int_0^t \lambda_2(s) ds}$$

Now, we proceed to determine approximate values for λ_1 and λ_2 . First we write the differential equation IV-3 in factored form,

$$\frac{d\lambda_i}{dt} = -(\lambda_i - r_1(t))(\lambda_i - r_2(t)), \quad (\text{IV-6})$$

and assume that

$$\lambda_1 = r_1 + \delta\lambda_1 \quad \left| \delta\lambda_1 \right| \ll \left| r_1 \right|, \quad \left| r_1 - r_2 \right|,$$

to obtain a linear differential equation

$$\delta \dot{\lambda} + (r_1 - r_2) \delta\lambda = -\dot{r}_1. \quad (\text{IV-7})$$

Also, assuming that $\left| \delta \dot{\lambda} \right| \ll \left| \dot{r}_1 \right|$, we solve IV-7 for $\delta\lambda$:

$$\delta\lambda = -\frac{\dot{r}_1}{r_1 - r_2} \quad (\text{IV-8})$$

or

$$\lambda_1 = r_1 - \frac{\dot{r}_1}{r_1 - r_2} \quad (\text{IV-9})$$

Now, to evaluate the consistency of the assumption regarding the smallness of $\delta\lambda$, we find that

$$\delta \dot{\lambda} = \frac{\dot{r}_1 (\dot{r}_1 - \ddot{r}_2)}{(r_1 - r_2)^2} - \frac{\ddot{r}_1}{r_1 - r_2}. \quad (\text{IV-10})$$

Therefore, if the quasi-steady roots, r_i , change with time in a reasonably linear fashion such that $\ddot{r}_1 \approx 0$, then

$$\delta \dot{\lambda} \sim (\delta\lambda)^2,$$

and we have a result that is correct to the first order in $\frac{\dot{r}_1}{r_1 - r_2}$. We may continue to develop further terms in a series $\frac{\dot{r}_1}{r_1 - r_2}$ in this fashion; however, the first-order approximation will suit our purposes.

The other solution is, by a similar procedure,

$$\lambda_2 = r_2 - \frac{\dot{r}_2}{r_2 - r_1} \quad (\text{IV-11})$$

Now, it is convenient to evaluate these effects by use of an extension of the root locus technique (Reference 33). From equation IV-6, with the assumption $\dot{r}_1 \cong \dot{\lambda}_1$, note that this assumption implies that $\delta\lambda$ is small, so that only a small portion of the root locus, the branch near the root under consideration, is valid. The differential equation IV-6 becomes an algebraic equation, since \dot{r}_1 is known:

$$\dot{r}_1 + (\lambda_1 - r_1)(\lambda_1 - r_2) = 0, \quad (\text{IV-12})$$

or in root locus form

$$\frac{\dot{r}_1}{(\lambda_1 - r_1)(\lambda_1 - r_2)} = -1, \quad (\text{IV-13})$$

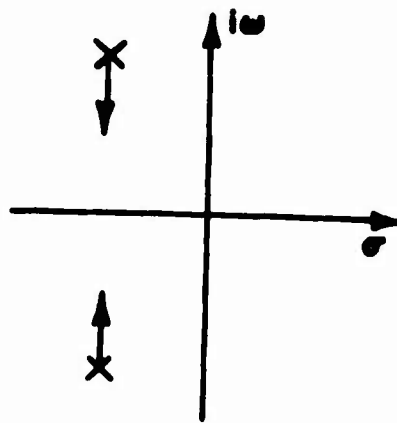
where \dot{r}_1 is considered as a gain.

We have one difference from the conventional root locus technique in that \dot{r}_1 may, in general, be complex, thus giving rise to angle conditions that are other than the conventional 0° and 180° conditions. The rules for constructing these loci are discussed in Appendix IV. Generally they are similar to conventional loci. If $\dot{r}_1 = R(t)_i e^{i\Phi(t)_i}$, then equation IV-13 becomes

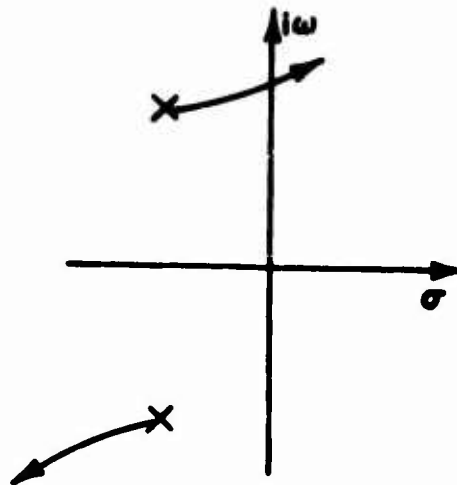
$$\frac{R(t)_i}{(\lambda_1 - r_1)(\lambda_1 - r_2)} = e^{i(\pi - \Phi_i)}. \quad (\text{IV-14})$$

The angle condition to be satisfied is $\pi - \Phi_i$. For the second-order system, if one root has a complex velocity, i. e., not parallel to the real axis, then the other root must have the conjugate velocity. The quasi-steady roots must always be conjugates, since the coefficients of the differential equation are real. The angle condition associated with the other root is therefore $\pi + \Phi_i$. Comparison of the angle condition for two roots with conjugate velocities indicates that the locus corresponding to the second-angle condition will be the locus corresponding to the first-angle condition reflected about the real axis.

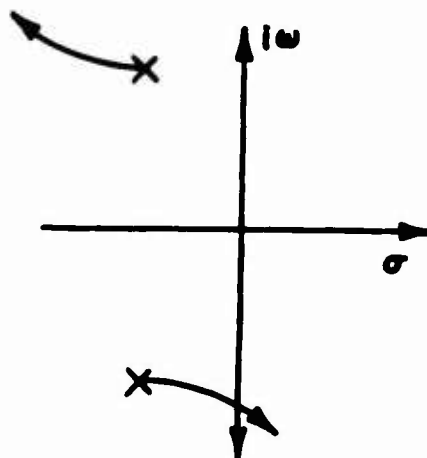
Let us take a simple case where the quasi-steady roots are moving parallel to the imaginary axis:



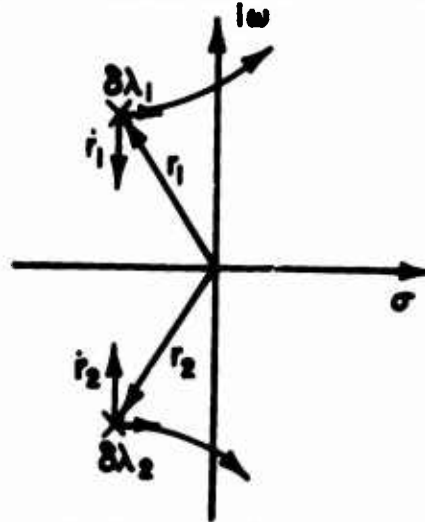
This is the quasi-steady locus. Then the unsteady locus at some instant of time for r_1 is:



where we only retain the upper branch as consistent with the assumption $\lambda_1 = \dot{r}_1$. Reflecting this locus about the real axis gives the locus for r_2 , where only the lower branch is consistent with the assumptions.



Thus, the unsteady locus at some instant of time is:



We then compute the gain in the usual fashion to locate the unsteady roots at the particular time instant of interest. A sample point is shown, as well as the quantities r_1 , r_2 and $\delta\lambda_1$, $\delta\lambda_2$. This locus then shows the effect discussed previously of a loss in apparent damping as the frequency decreases. Recall that only the approximately straight portion of the locus is valid within the limits of the approximation.

We then construct these loci about sufficient quasi-steady points, such that we can draw the unsteady locus and the "roots" to be used in the approximate solution.

The magnitude of the deviation of the unsteady locus from the quasi-steady locus may be estimated by taking the magnitude of the terms in equation IV-14:

$$\frac{R(t)}{|\lambda_1 - r_1| |\lambda_1 - r_2|} = 1 \quad (\text{IV-15})$$

Linearizing the denominator of IV-15,

$$|\lambda_1 - r_1| = |\delta\lambda|$$

$$|\lambda_1 - r_2| \cong |r_1 - r_2|$$

and

$$|r_1 - r_2| = |2\omega_{QS}|,$$

where ω_{QS} is the frequency of the quasi-steady solution.

So we have

$$\frac{|\dot{r}|}{|\delta\lambda| 2\omega_{QS}} = 1 ,$$

$$|\delta\lambda| = \frac{|\dot{r}|}{2\omega_{QS}} . \quad (IV-16)$$

The fractional change of $\delta\lambda$ is:

$$\left| \frac{\delta\lambda}{r} \right| = \frac{|\dot{r}|}{2\omega_{QS}|r|} . \quad (IV-17)$$

That is, the fractional change in the magnitude of the "root" due to unsteady effects is the fractional change in the root per unit time divided by twice the instantaneous frequency (the root spacing on the complex plane).

The results are exactly the same as obtained from the asymptotic solution.

The n^{th} -Order Equation

The approach for the second-order equation may be generalized to yield approximate solutions to an n^{th} -order linear differential equation with slowly varying coefficients. The precise meaning of slowly varying coefficients will be defined by our result. Given an n^{th} -order differential equation,

$$\frac{d^n x}{dt^n} + a_{n-1}(t) \frac{d^{n-1} x}{dt^{n-1}} + a_{n-2}(t) \frac{d^{n-2} x}{dt^{n-2}} \dots + a_0(t) x = 0 . \quad (IV-18)$$

We assume a solution of the form

$$x(t) = \sum_{i=1}^n C_i e^{\int_0^t \lambda_i(s) ds} \quad (IV-19)$$

The n^{th} -order derivative of x , considering only one λ_i , may be written as:

$$\begin{aligned} \frac{d^n x}{dt^n} = & \lambda_i^n + (n)(n-1) \left(\frac{\dot{\lambda}_i \lambda_i^{n-2}}{2!} \right) + (n)(n-1)(n-2) \left(\frac{\ddot{\lambda}_i \lambda_i^{n-3}}{3!} \right) + \\ & (n)(n-1)(n-2)(n-3) \left(\frac{\ddot{\lambda}_i}{4!} + \frac{\dot{\lambda}_i^2}{4 \cdot 2} \right) \lambda_i^{n-4} + \dots \end{aligned} \quad (IV-20)$$

(Terms involving higher derivatives, and products of derivatives of λ_i)

Substitution of these derivatives into the differential equation IV-18 yields the following nonlinear differential equation for λ , of order $n-1$:

$$\Delta + \frac{1}{2!} \frac{\partial^2 \Delta}{\partial \lambda^2} \dot{\lambda}_i + \frac{1}{3!} \frac{\partial^3 \Delta}{\partial \lambda^3} \ddot{\lambda}_i + \frac{1}{4!} \frac{\partial^4 \Delta}{\partial \lambda^4} (\ddot{\lambda}_i + 3\dot{\lambda}_i^2) \dots$$

$$+ (\text{Terms involving higher derivatives, and products of derivatives of } \lambda_i) = 0$$

(IV-21)

where

$$\Delta = \lambda_i^n + a_{n-1}(t) \lambda_i^{n-1} + a_{n-2}(t) \lambda_i^{n-2} \dots a_0(t) \quad (IV-22)$$

In the development that follows it is useful to note that the function Δ and its derivatives may be written in factored form as:

$$\Delta = \prod_{j=1}^n (\lambda_i - r_j) \quad (IV-23a)$$

$$\frac{\partial \Delta}{\partial \lambda} = \prod_{j=1}^n (\lambda_i - r_j) \left(\sum_{k=1}^n \frac{1}{(\lambda_i - r_k)} \right) \quad (IV-23b)$$

$$\frac{1}{2!} \frac{\partial^2 \Delta}{\partial \lambda^2} = \prod_{j=1}^n (\lambda_i - r_j) \left(\sum_{k, \ell=1}^n \frac{1}{(\lambda_i - r_k)(\lambda_i - r_\ell)} \right) \quad k \neq \ell. \quad (IV-23c)$$

Using the above relationships we linearize the differential equation IV-21 by assuming that each unsteady root, λ_i , is near to a corresponding quasi-steady root, r_i , and that the quasi-steady roots are isolated. Therefore,

$$\lambda_i = r_i + \delta\lambda_i \quad \text{where} \quad |\delta\lambda_i| \ll |r_i|, \quad |r_i - r_j|.$$

Substituting this relationship into the expressions IV-23 and retaining only terms up to the first order in $\delta\lambda_i$, we obtain

$$\Delta = \Delta \left|_{\lambda_i=r_i} + \frac{\partial \Delta}{\partial \lambda} \right|_{\lambda=r_i} \delta\lambda_i = \sum_{j=1}^n (r_i - r_j) \delta\lambda_i, \quad i \neq j. \quad (\text{IV-24a})$$

$$\begin{aligned} \frac{1}{2!} \frac{\partial^2 \Delta}{\partial \lambda^2} &= \frac{1}{2!} \left(\frac{\partial^2 \Delta}{\partial \lambda^2} \right)_{\lambda_i=r_i} + \frac{\partial^3 \Delta}{\partial \lambda^3} \delta\lambda_i = \\ &= \sum_{j=1}^n (r_i - r_j) \left(\sum_{k, \ell=1}^n \frac{1}{(r_i - r_k)(r_i - r_\ell)} + \sum_{k, \ell, p=1}^n \frac{1}{(r_i - r_k)(r_i - r_\ell)(r_i - r_p)} \delta\lambda_i \right). \\ &\quad i \neq j; k \neq \ell \neq p \end{aligned} \quad (\text{IV-24b})$$

Denote the distance on the complex plane from the i^{th} quasi-steady root to the j^{th} quasi-steady root by d_{ij} . To nondimensionalize expressions IV-24, we designate as the minimum of the distances d_{ij} and the distances of any root from the origin, r_i , during the time interval of interest, as d_s , and divide all distances involved by d_s , and call the non-dimensional distances D_{ij} , R_i , $\delta\Lambda_i$. The order of magnitude of these three quantities is taken to be:

$$D_{ij}, R_i \quad 0(1),$$

$$\delta\Lambda_i \quad 0(\epsilon).$$

The above expressions become

$$\Delta = d_s^n \sum_{j=1}^n (D_{ij}) \delta\Lambda_i, \quad j \neq i \quad (\text{IV-25a})$$

$$\frac{1}{2} \frac{\partial^2 \Delta}{\partial \lambda^2} = d_s^n \prod_{j=1}^n (D_{ij}) \left(\sum_{j=1}^n \frac{1}{D_{ij}} + \sum_{j,k=1}^n \frac{1}{D_{ij} D_{ik}} \delta \Lambda_i \right) \quad \begin{matrix} j \neq i \\ j \neq k \end{matrix} \quad (\text{IV-25b})$$

Then, we write the differential equation IV-21 in these terms, taking as the characteristic time the reciprocal of the maximum value of $\delta \lambda_i$, $\delta \lambda_m$. Recall that although we have referred to $\delta \lambda_i$ as a distance on the complex plane, the units of $\delta \lambda_i$ are sec^{-1} . Let $\tau = \delta \lambda_m t$, and for equation IV-21 after division by $d_s^n \prod_{j=1}^n (D_{ij})$, we have

$$\begin{aligned} & \delta \Lambda_i + \left(\sum_{j=1}^n \frac{1}{D_{ij}} + \sum_{j,k=1}^n \frac{1}{D_{ij} D_{ik}} \delta \Lambda_i \right) \left(\frac{dR_i}{d\tau} + \frac{d(\delta \Lambda_i)}{d\tau} \right) \delta \Lambda_m \\ & \left(\sum_{j,k=1}^n \frac{1}{D_{ij} D_{ik}} + \sum_{j,k,\ell=1}^n \frac{1}{D_{ij} D_{ik} D_{i\ell}} \delta \Lambda_i \right) \left(\frac{d^2 R_i}{d\tau^2} + \frac{d^2(\delta \Lambda_i)}{d\tau^2} \right) (\delta \Lambda_m)^2 \\ & \left(\sum_{j,k,\ell=1}^n \frac{1}{D_{ij} D_{ik} D_{i\ell}} + \sum_{j,k,\ell,p=1}^n \frac{1}{D_{ij} D_{ik} D_{i\ell} D_{ip}} \delta \Lambda_i \right) \\ & \left[\left(\frac{d^3 R_i}{d\tau^3} + \frac{d^3(\delta \Lambda_i)}{d\tau^3} \right) (\delta \Lambda_m)^3 + 3 \left(\frac{dR_i}{d\tau} + \frac{d(\delta \Lambda_i)}{d\tau} \right)^2 (\delta \Lambda_m)^2 \right] = 0. \end{aligned}$$

$j \neq i$
 $j \neq k$
 $j \neq \ell$
 $j \neq p$

(IV-26)

Neglecting $\delta \Lambda_i$ compared to 1, and the derivatives of $\delta \Lambda_i$ compared to the derivatives of R_i , we obtain

$$\delta\Lambda_i \cong - \left(\sum_{j=1}^n \frac{1}{D_{ij}} \right) \left(\frac{dR_i}{d\tau} \right) (\delta\Lambda_m) - \left(\sum_{j,k=1}^n \frac{1}{D_{ij} D_{ik}} \right) \left(\frac{d^2 R_i}{d\tau^2} \right) (\delta\Lambda_m)^2 -$$

$$\left(\sum_{j,k,l=1}^n \frac{1}{D_{ij} D_{ik} D_{il}} \right) \left(\frac{d^3 R_i}{d\tau^3} \right) (\delta\Lambda_m)^3 + 3 \left(\frac{dR_i}{d\tau} \right)^2 (\delta\Lambda_m)^2. \quad (\text{IV-27})$$

Since $\delta\Lambda_i$ and $\delta\Lambda_m$ are assumed small, and the derivatives of R_i in this time scale are the order of 1, the approximate solution of equation IV-21, retaining only first-order terms in IV-27, is:

$$\delta\Lambda_i \cong - \left(\sum_{j=1}^n \frac{1}{D_{ij}} \right) \frac{dR_i}{d\tau} (\delta\Lambda_m). \quad (\text{IV-28})$$

The basic requirement for this approximate solution to hold is then that

$$\frac{dR_i}{d\tau} = O(1).$$

This may be expressed in the time scale of the dynamics of the system $\left(\frac{1}{d_s} \right)$, as

$$\frac{1}{\delta\Lambda_m} \frac{dR_i}{d(d_s t)} = O(1),$$

and therefore

$$\frac{dR_i}{d(d_s t)} = O(\epsilon), \quad \frac{d^n R_i}{d(d_s t)^n} = O(\epsilon^n).$$

Thus we require that the fractional change in the quasi-steady roots is small in a time scale based on the smallest of the spacing of the quasi-steady roots or the distance of the quasi-steady roots from the origin on the complex plane.

The phrase "slowly varying coefficients" means that $\frac{d\left(\frac{R_i}{d_s}\right)}{d(d_s t)}$ is small.

The relationships between R_i and its derivatives will generally be true for monotonically varying roots as time becomes large.

The solution, IV-28, may be expressed in root locus form, convenient for calculations from equation IV-21:

$$\frac{\frac{1}{2} \frac{\partial^2 \Delta}{\partial \lambda^2} \dot{r}_i}{\Delta} = -1, \quad (\text{IV-29})$$

noting that only a small part of the locus will be valid, due to the assumptions made in obtaining the approximate solution.

The form of the result, IV-28, indicates that the quasi-steady effects tend to increase with the order of the system because of the presence of the term involving the summation of the reciprocals of the root spacings. The time-varying effects are dominated by the lowest frequencies, as well as by the close proximity of roots.

The Third-Order Equation

We now apply the n^{th} -order result to a third-order equation. Given the differential equation

$$\frac{d^3 x}{dt^3} + a(t) \frac{d^2 x}{dt^2} + b(t) \frac{dx}{dt} + c(t) x = 0, \quad (\text{IV-30})$$

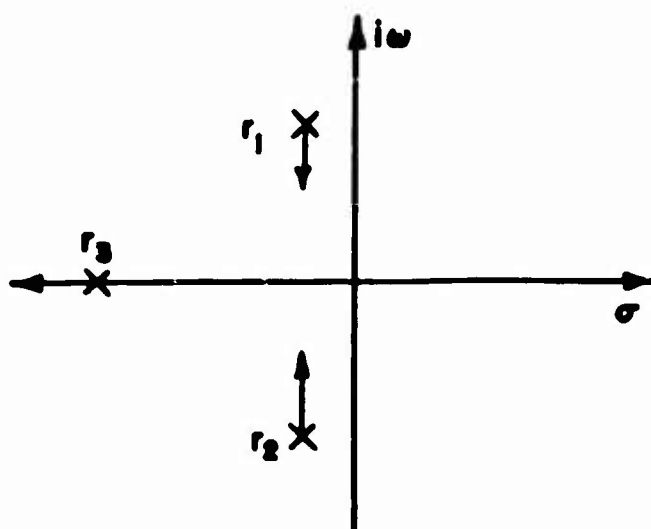
application of equation IV-29 gives

$$\frac{\frac{1}{2} \frac{\partial^2 \Delta}{\partial \lambda^2} \dot{r}_i}{\Delta} = \frac{(3\lambda + a) \dot{r}_i}{(\lambda - r_1)(\lambda - r_2)(\lambda - r_3)} = -1. \quad (\text{IV-31})$$

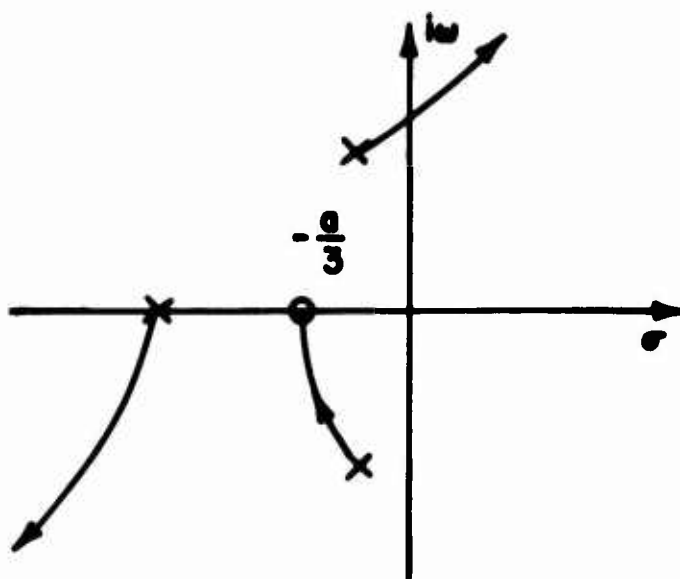
The poles for this locus are the quasi-steady roots, r_i ; there is one zero at $-\frac{a}{3}$, and, in general, a complex gain, \dot{r}_i .

For all real quasi-steady roots, the gain will be real. For a complex pair and a real root, two loci will have complex gains, similar to the second-order case discussed, where only the branch associated with the particular gain is retained. The third locus corresponds to the real root, where only the branch originating at the real root need be considered, and has a real gain.

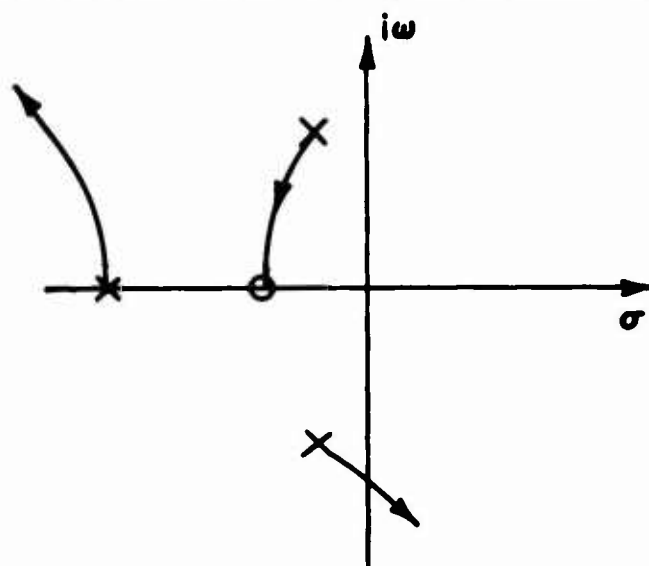
Let us sketch these loci for a particular case. Consider a third-order system where the quasi-steady roots consist of a complex pair and a real root, the complex pair moving towards the real axis and the real root moving away from the imaginary axis.



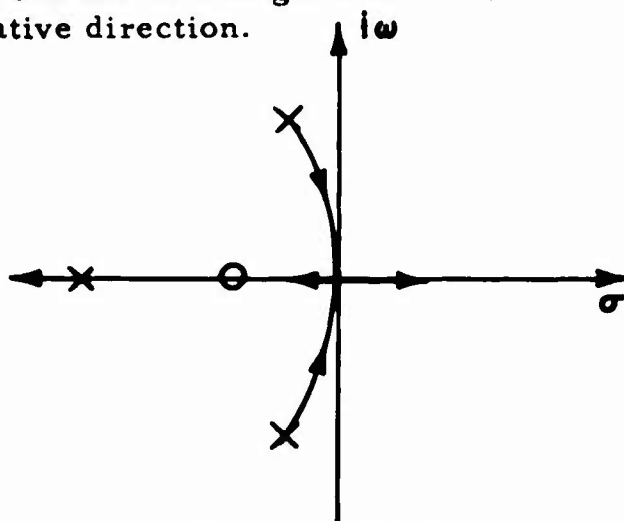
The angle condition for the r_1 locus is $270^\circ - \pi$ and we must consider the zero at $-\frac{a}{3}$. Thus the locus is:



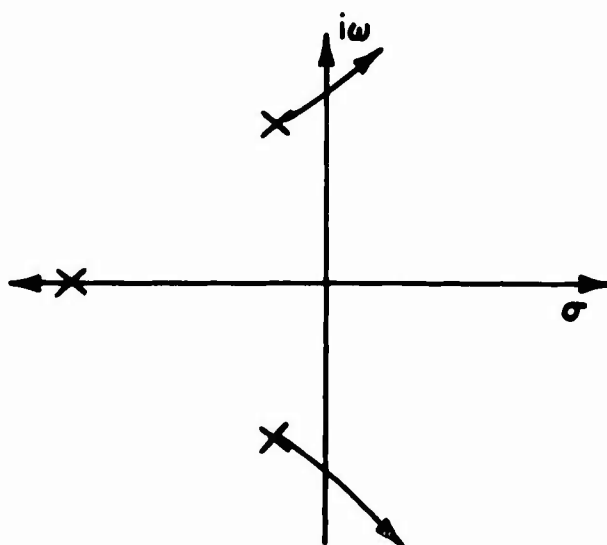
The locus for r_2 is the reflection of this locus about the real axis:



The locus associated with the movement of the real root is a conventional locus drawn for the zero angle condition, since the real root is moving in the negative direction.



Thus, the unsteady locus for this example is



where only the branch associated with each particular quasi-steady root has been retained.

The magnitude of the deviation is computed in a conventional way, recalling that we use the root velocity associated with the particular root; i. e. , there will be a different gain associated with the real root branch from that associated with the complex branches.

The importance of the root velocity and root locations in causing deviations from the quasi-steady locus may be seen directly by application of equation IV-28. For this example, equation IV-28, in dimensional form, becomes:

$$\delta\lambda_1 = - \left(\frac{1}{r_1 - r_2} + \frac{1}{r_1 - r_3} \right) \dot{r}_1 . \quad (\text{IV-32})$$

Comparison of equation IV-32 with IV-8 indicates an additional effect on the oscillatory motion in the third-order case, due to the factor $\left(\frac{1}{r_1 - r_2} + \frac{1}{r_1 - r_3} \right)$. If the real quasi-steady root, r_3 , is a considerably faster mode than the mode corresponding to the complex pair, r_1, r_2 , then the second term in parenthesis is approximately zero, and we have the same result as the second-order case. When the mode associated with the real root is slower than the frequency of the oscillatory mode, the magnitude of the deviation of the unsteady locus from the quasi-steady locus will be increased, compared to considering the same complex pair as an isolated second-order system.

It should also be noted that for the specific third-order case where a real quasi-steady root is constant we can reduce the third-order time-varying equation to a second-order time-varying equation by the following transformation.

Given a third-order equation, with $-r_1$ a real constant quasi-steady root, the differential equation, IV-30, becomes

$$\frac{d^3x}{dt^3} + [r_1 + d(t)] \frac{d^2x}{dt^2} + [r_1 d(t) + f(t)] \frac{dx}{dt} + r_1 f(t) x = 0 . \quad (\text{IV-33})$$

The transformation

$$x = e^{-r_1 t} \int_0^t y(s) ds \quad (\text{IV-34})$$

transforms (IV-33) to

$$\frac{d^2 y}{dt^2} + [2r_1 + d(t)] \frac{dy}{dt} + [f(t) + r_1 d(t) + r_1^2] y = 0. \quad (\text{IV-35})$$

We can determine the approximate solution to this second-order differential equation and then integrate the result to obtain x . This simple case also indicates that the importance of the unsteady effects will increase as the order of the differential equation increases. A further amplitude modification will result from a varying frequency when y is integrated to obtain x .

This might have been expected from our earlier discussion where we saw that the quasi-steady solution was exact for the first-order equation, but only a rough approximation to the second-order equation.

To be specific, consider the nature of the solution when $r_1 = 0$, $d(t) = 0$. Assuming that the asymptotic approximation applies to the solution of the differential equation IV-35, then

$$y = \frac{C_1}{f^{\frac{1}{4}}} \cos \left(\int_0^t f^{\frac{1}{2}} ds + \Phi \right); \quad (\text{IV-36})$$

x is then determined from equation IV-34,

$$x = C_1 \int_0^t f^{-\frac{1}{4}} \cos \left(\int_0^s f^{\frac{1}{2}} dn + \Phi \right) ds. \quad (\text{IV-37})$$

The integral, IV-37, may be approximated by

$$x \cong \frac{C_1}{f^{\frac{3}{4}}} \sin \left(\int_0^t f^{\frac{1}{2}} dn + \Phi \right), \quad (\text{IV-38})$$

when f is slowly varying, as consistent with the asymptotic approximation.

The same result will be obtained by applying the root locus approach directly to the third-order equation.

We then see that in the third-order system similar effects to those obtained in a second-order system result. That is, a decreasing frequency causes an apparent loss in damping. We obtain an additional effect from the third "root," the magnitude depending upon the location of this real root with respect to the complex pair.

The maximum loss in damping is associated with the case in which the magnitude of the real parts of all the roots are the same, as shown by the above example.

Summary

An approximate solution to a linear differential equation of order n , with time-varying coefficients

$$\frac{d^n x}{dt^n} + a_{n-1}(t) \frac{d^{n-1} x}{dt^{n-1}} + a_{n-2}(t) \frac{d^{n-2} x}{dt^{n-2}} + \dots + a_0 x = 0$$

may be expressed in the following form:

$$x(t) = \sum_{i=1}^n C_i e^{\int_0^t (\lambda_i + \delta\lambda_i) ds}$$

where

$$\delta\lambda_i = - \left(\sum_{j=1}^n \frac{1}{(r_i - r_j)} \right) \dot{r}_i \quad i \neq j$$

and the r_i are roots of the equation

$$r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \dots + a_0 = 0.$$

Thus for a given \dot{r} , as the order of the system increases, the unsteady effects increase, as shown by the coefficient of \dot{r}_i in the equation for $\delta\lambda_i$. The magnitude of the effects depends upon the separation of the roots on the complex plane $(r_i - r_j)$.

Consider the change in the root $\delta\lambda_i$ compared to r_i .

$$\frac{\delta\lambda_i}{r_i} = - \left(\sum_{j=1}^n \frac{1}{(r_i - r_j)} \right) \frac{\dot{r}_i}{r_i} \quad i \neq j$$

The change in the root r_i due to unsteady effects is therefore the sum of the fractional change in the root r_i per unit time divided by each of the root spacings on the complex plane. The restriction on the approximate solution phrased in these terms is that the fractional change in the roots r_i per unit time must be small compared to the root spacings.

Specific examples of the application of this result are given in Chapter V.

The approximate methods presented in this chapter are readily applied to the investigation of the following problem. Determine the dynamics of a system as it passes continuously through or near to a series of equilibrium states, when the frozen dynamics of the system are known at these equilibrium states. The changes in the system dynamics are considered to arise from two sources: the first is the appearance of new terms in the uncoupled differential equations describing each variable in a coupled system, and the second is the direct effect of the coefficients changing with time. The former effect is referred to as a quasi-steady effect, and the latter as an unsteady effect. The quasi-steady effects are taken into account directly in the computation of the r_i , and the unsteady effects are taken into account by the approximate solution presented above. These two effects may be interpreted as distorting the frozen locus of roots on the complex plane.

CHAPTER V

APPLICATION TO AIRCRAFT DYNAMICS IN UNSTEADY FLIGHT

Before proceeding to the development of aircraft equations of motion, some preliminary considerations regarding the nature and implications of the approach to the problem are useful. As anticipated, the resulting equations of motion describing the dynamics of aircraft in unsteady flight will be linear differential equations with time-varying coefficients. Let us consider, in general, how linear time-varying differential equations arise in physical problems.

One way in which the description of a system in terms of linear equations with time-varying coefficients arises could be referred to as a true linear problem, where the coefficients in the differential equation change due to external influences on the system. A simple example is a pendulum with its length being varied continuously.

A second way that systems described by linear variable coefficient equations arise is in investigating the motion of a nonlinear system near a state of motion rather than a state of equilibrium. Since we are investigating a state of motion near another state of motion, we may linearize the equations of motion by a perturbation approach, about the original state of motion. When the reference state is changing with time, we obtain linear variable coefficient equations to describe the perturbed motion. Note that if the original system is linear, both of these situations will be described by the same differential equation. Again the pendulum serves as a simple example (Reference 13). Consider the large amplitude oscillation of a pendulum, described by the nonlinear differential equation

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0. \quad (V-1)$$

We consider an established state of motion resulting from the initial conditions

$$\begin{aligned} \theta(0) &= \alpha, \\ \dot{\theta}(0) &= \beta. \end{aligned} \quad (V-2)$$

We call the solution to equation V-1 with initial conditions V-2, $m(t)$, and wish to determine the differential equation describing the difference between this motion and some neighboring motion arising from slightly different initial conditions,

$$\theta_1(0) = \alpha + \delta\alpha,$$

$$\dot{\theta}_1(0) = \beta + \delta\beta. \quad (V-3)$$

We call the solution to equation V-1, with initial conditions V-3

$$n(t) = m(t) + \delta\theta \quad (V-4)$$

where $\delta\theta$ is small. Substituting V-4 into V-1,

$$\ddot{\delta\theta} + \ddot{m}(t) + \frac{g}{l} \sin(m(t) + \delta\theta) = 0. \quad (V-5)$$

Expanding the restoring force term in a Taylor series, we obtain for small $\delta\theta$,

$$\ddot{\delta\theta} + \ddot{m}(t) + \frac{g}{l} [\sin(m(t)) + \cos(m(t)) \delta\theta] = 0. \quad (V-6)$$

Using the fact that $m(t)$ is a solution to V-1, the differential equation describing the perturbed motion is:

$$\ddot{\delta\theta} + \frac{g}{l} (\cos m(t)) \delta\theta = 0. \quad (V-7)$$

Thus, since $m(t)$ is a function of time, equation V-7, linearized about a state of motion, is a linear time-varying equation.

In studying the dynamics of aircraft in flight, with changing velocity, the time-varying nature of the coefficients is a result both of linear or external effects and the linearization of nonlinear equations about a variable state of motion. Control motions and changes in power settings, for example, are linear effects.

If the coefficients of the linearized differential equation are changing rapidly with time, we must examine carefully the validity of the linearized approach as seen from the following considerations. The form of the linearized differential equation V-7 is:

$$\ddot{\delta\theta} + \frac{\partial f}{\partial \theta} \delta\theta = 0, \quad (V-8)$$

where f is the nonlinear restoring force term ($\frac{g}{l} \sin \theta$ in this case).

The importance of nonlinear effects are indicated by the rate of change of $\frac{\partial f}{\partial \theta}$ with θ times the size of the perturbation; i. e., the

linearization assumption implies that

$$\frac{\frac{\partial^2 f}{\partial \theta^2} \frac{\delta \theta}{2}}{\frac{\partial f}{\partial \theta}} \quad (V-9)$$

is small. The importance of time-varying effects is shown by the parameter discussed in Chapter III, which here takes the form:

$$\frac{\frac{\partial^2 f}{\partial \theta^2} \frac{d\theta_p}{dt}}{\frac{\partial f}{\partial \theta} \sqrt{\frac{\partial f}{\partial \theta}}} \quad (V-10)$$

We have assumed that f is not an explicit function of time; i. e., we have no truly linear effects.

Thus, as the parameter V-10 becomes large, the time-varying effects become important. The cause of the parameter V-10 increasing in size may or may not cause an increase in the size of the parameter V-9. If V-10 becomes large due to increases in $\frac{d\theta_p}{dt}$, i. e., rate of change of the reference displacement with time, then we may have large time-varying effects, and still the linearization is valid. If, however, V-10 becomes large due to $\frac{\partial f}{\partial \theta} \rightarrow 0$, V-9 becomes large as well, and it may be that linearization is not valid. In particular, it appears that care must be taken in studying nonlinear systems as linearized time-varying systems near the zeros of important coefficients.

Considerations relating to the above questions with regard to the stability derivatives of aircraft and the linearity of terms are discussed in Appendices II and III.

For an aircraft, the time-varying approach therefore is expected to be generally more nearly valid when changes in dynamics arise from rapid changes in flight condition of the vehicle. It would appear that a nonlinear approach may be necessary when the stability derivatives themselves are rapidly changing with flight condition. Recall that this does not apply to the influence of control and power settings which are truly linear time-varying effects, but does apply to such effects as the rapid change of the stability derivatives with flight velocity.

A. Development of Linearized Equations of Motion of Aircraft

We now formulate the linearized equations of motion to study the specific problem of the effect of varying flight velocity on the dynamic stability and response characteristics of aircraft. Dynamic motions will be restricted to the longitudinal plane of symmetry.

We write the equations of motion with respect to an axis system fixed to the body of the aircraft. The X axis is roughly horizontal, the Z axis is vertical, and the origin of the axis system is at the center of gravity of the airplane. The equations of motion may be written, then, in the following conventional form (Reference 14):

$$\begin{aligned}\frac{du}{dt} + wq + g \sin \theta &= X(u, w, q, \delta_T, \delta_E, i_w) \\ \frac{dw}{dt} - uq - g \cos \theta &= Z(u, w, q, \delta_T, \delta_E, i_w) \\ \frac{dq}{dt} &= M(u, w, q, \delta_T, \delta_E, i_w) \\ \frac{d\theta}{dt} &= q\end{aligned}\tag{V-11}$$

The terms on the left-hand side are the gravity and inertia forces. The terms on the right-hand side are the aerodynamic forces per unit mass and moments per inertia. u and w are the variables describing the motion of the vehicle with respect to the X and Z axes, respectively, moving with angular velocity q . δ_T , δ_E , and i_w are the parameters by which the pilot exerts control over the vehicle. In the specific instance of a tilt-wing aircraft, for example, δ_T represents propeller blade pitch, δ_E the longitudinal pitching moment control, and i_w the wing tilt angle. A conventional airplane normally has only two controls (δ_T and δ_E).

It may be noted that no new aerodynamic terms are required for the case of nonsteady flight, since we assume that unsteady aerodynamic effects are not important. The stability derivatives themselves, i. e., the terms in the Taylor series expansion of the aerodynamic forces, may depend upon the nature of the maneuver being performed. These points are discussed in detail in Appendices II and III.

The set of differential equations, V-11, is nonlinear and, if the control settings are not constant, is also time varying.

The equations are nonlinear, due to the nature of the aerodynamic force and moment dependence upon the flight variables, the inertia terms arising from use of a Eulerian axis system, and the gravity terms.

A particular set of solutions to equations V-11 represents large-scale maneuvers in which the airplane is not allowed to rotate, or in which the rotation is prescribed. Then, the airplane is essentially considered as a point mass, the pilot having primary control over the motion of the point mass with two of the three controls. The pitching moment equation provides the information as to what must be done with the third control to prevent the rotation or to attain the prescribed rotation. We might imagine, in considering these maneuvers, an airplane with zero inertia and angular damping such that it responds instantaneously to control motions.

To be specific, the accelerated flight of an aircraft is considered in the following way. Certain prescribed control motions are made in order to perform a maneuver. Then, imagine this maneuver being repeated with exactly the same control motions, but now the aircraft encounters disturbances. We wish to investigate the response of the aircraft to these disturbances while performing the prescribed maneuver. It may be imagined that the control motions are programmed to perform the maneuver. This model has its counterpart in the stick fixed response of an airplane in steady flight. In many instances, it is possible that these basic control motions are of considerable importance in the piloting task, but they will not be considered here.

As examples of these motions for flight vehicles, the following may be enumerated.

1. Steady, level flight
2. Accelerating or decelerating level flight
3. The trajectory of a ballistic missile
4. The transition of a VTOL aircraft from hover to forward flight

To obtain the relationships between the flight variables and the control settings for examples 1 and 2 is almost trivial for a conventional aircraft, since the control over the movements of the center of gravity are approximately uncoupled. The pilot has only two controls: the thrust of the power plant which directly controls acceleration, and the elevator angle which controls rotation of the airplane and indirectly controls the flight velocity, maintaining the proper relationship between flight velocity and angle of attack. Solutions to example 3 are studied in References 8 and 9.

Item 4 is a more complicated dynamics problem, due to the strong influence of engine thrust on vertical as well as horizontal force. This interrelationship causes deviations from steady flight conditions when the airplane is accelerating or decelerating. In order to obtain an estimate of the magnitude of the variation in the flight conditions encountered during this dynamic maneuver compared to steady flight, two different VTOL aircraft are considered in Appendix I. A tilt-jet and a tilt-wing, two types of aircraft that fly in the speed range where an appreciable portion of the vertical force is developed by thrust, are considered. The former performs a fairly rapid transition, and the latter a relatively slow transition. The control required to trim is not considered in this analysis, and the transitions are restricted to level flight for simplicity. A highly simplified model is used to gain an understanding of the essential features of the motion.

Having once determined the nature of this reference motion for a conventional or VTOL aircraft, we can proceed to investigate the basic question of interest. That is, if the airplane encounters a small disturbance while performing this evolution, we wish to predict this disturbed motion. We assume that the pilot takes no action to counter the disturbance, and only investigate the nature of the uncontrolled response.

To obtain results of a general nature, rather than solving equations V-11 as they stand, using machine methods, we will linearize the equations about a prescribed path such that the transition maneuver will correspond to the trivial solution of the perturbation equations.

The first step is to linearize the aerodynamic forces by expansion in a Taylor series about a prescribed point on the path. This approach has been notably successful in the past as a means of treating the aerodynamic forces of conventional aircraft during a response (Reference 14). It is suspected that this linearization is valid for VTOL aircraft at low speeds in steady flight, although further experimental data is required before linearization can be applied with complete confidence. These questions in relation to VTOL aircraft are discussed in some detail in Appendix III.

The equations of motion are:

$$\begin{aligned}\frac{du}{dt} + wq + g \sin \theta &= X(u, w, \delta_T, i_w) \\ \frac{dw}{dt} - uq - g \cos \theta &= Z(u, w, \delta_T, i_w) \\ \frac{dq}{dt} &= M(u, w, q, \delta_T, \delta_E, i_w) \\ \frac{d\theta}{dt} &= q\end{aligned}\tag{V-12}$$

where we have neglected the dependence on X and Z on q and δ_E for simplicity. These terms usually do not exert a strong influence on the motion.

Given the reference motion, described by $U_P, W_P, \delta_{T_P}, i_{w_P}, q_P, \delta_{T_P}$ at some instant of time, we assume that in the neighborhood of these values we may describe the variation of the aerodynamic forces by retaining only the first-order terms in a Taylor series expansion.

$$X + \Delta X = X(U_P, W_P, \delta_{T_P}, i_{w_P}) + \frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial \delta_T} \Delta \delta_T + \frac{\partial X}{\partial i_w} \Delta i_w$$

$$Z + \Delta Z = Z(U_P, W_P, \delta_{T_P}, i_{w_P}) + \frac{\partial Z}{\partial u} \Delta u + \frac{\partial Z}{\partial w} \Delta w + \frac{\partial Z}{\partial \delta_T} \Delta \delta_T + \frac{\partial Z}{\partial i_w} \Delta i_w$$

$$M + \Delta M = M(U_P, W_P, q_P, \delta_{T_P}, \delta_{E_P}, i_{w_P}) + \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial q} \Delta q +$$

$$\frac{\partial M}{\partial \delta_T} \Delta \delta_T + \frac{\partial M}{\partial \delta_E} \Delta \delta_E + \frac{\partial M}{\partial i_w} \Delta i_w. \quad (V-13)$$

The coefficients of Δu , Δw , and Δq are called stability derivatives, and the coefficients of $\Delta \delta_T$, $\Delta \delta_E$, and Δi_w are control parameters.

We may now linearize the terms on the left-hand side, again assuming small deviations from the reference motion:

$$\frac{dU_P}{dt} + \frac{d\Delta u}{dt} + W_P \Delta q + q_P \Delta w + g \sin \theta_P + g \cos \theta_P \Delta \theta = X + \Delta X$$

$$\frac{dW_P}{dt} + \frac{d\Delta w}{dt} - U_P \Delta q - q_P \Delta u - g \cos \theta_P + g \sin \theta_P \Delta \theta = Z + \Delta Z$$

(V-14)

$$\frac{dq_P}{dt} + \frac{d\Delta q}{dt} = M + \Delta M.$$

We consider near level flight where θ_P is small, $W_P \approx 0$, $q_P \approx 0$, $\theta_P \approx 0$, so that the following linearized equations result in:

$$\frac{dU}{dt} \mathbf{P} + \frac{d\Delta u}{dt} + g \Delta \theta = X_{\mathbf{P}} + \frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial i_w} \Delta i_w + \frac{\partial X}{\partial \delta_T} \Delta \delta_T$$

$$\frac{d(\Delta w)}{dt} - U_{\mathbf{P}} \Delta q - g = Z_{\mathbf{P}} + \frac{\partial Z}{\partial u} \Delta u + \frac{\partial Z}{\partial w} \Delta w + \frac{\partial Z}{\partial i_w} \Delta i_w + \frac{\partial Z}{\partial \delta_T} \Delta \delta_T \quad (\text{V-15})$$

$$\frac{d(\Delta q)}{dt} = M_{\mathbf{P}} + \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial q} \Delta q + \frac{\partial M}{\partial \delta_T} \Delta \delta_T + \frac{\partial M}{\partial \delta_E} \Delta \delta_E +$$

$$\frac{\partial M}{\partial i_w} \Delta i_w$$

Now again, we assume that at any constant reference condition ($U_{\mathbf{P}} = \text{constant}$, $W_{\mathbf{P}} = \text{constant}$), equations V-15 are valid. And, as the airplane moves continuously through various sets of reference conditions we assume the equations still to be valid. That is, no additional assumptions are necessary to linearize the equations about a varying set of flight conditions if the linearization of the equations is valid about each flight condition encountered. As the airplane changes flight condition, the coefficients of the Taylor expansions of the aerodynamic forces, i. e., the stability derivatives, which are functions of the flight condition, become functions of time.

When there are no deviations from the prescribed path, all of the perturbation quantities are zero, and we have

$$\begin{aligned} \frac{dU}{dt} \mathbf{P} &= X_{\mathbf{P}} \\ -g &= Z_{\mathbf{P}} \\ 0 &= M_{\mathbf{P}} \end{aligned} \quad (\text{V-16})$$

Thus we may eliminate these terms from V-15, resulting in:

$$\frac{d(\Delta u)}{dt} + g \Delta \theta = \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial \delta_T} \Delta \delta_T + \frac{\partial X}{\partial i_w} \Delta i_w$$

$$\frac{d(\Delta w)}{dt} - U_P \Delta q = \frac{\partial Z}{\partial w} \Delta w + \frac{\partial Z}{\partial u} \Delta u + \frac{\partial Z}{\partial \delta_T} \Delta \delta_T + \frac{\partial Z}{\partial i_w} \Delta i_w \quad (V-17)$$

$$\frac{d(\Delta q)}{dt} = \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial q} \Delta q + \frac{\partial M}{\partial i_w} \Delta i_w + \frac{\partial M}{\partial \delta_E} \Delta \delta_E + \frac{\partial M}{\partial \delta_T} \Delta \delta_T,$$

where the stability derivatives are functions of the path variables, and consequently, functions of time; e. g. ,

$$\frac{\partial X}{\partial w} = \frac{\partial X}{\partial w} (U_P(t), W_P(t), \delta_{TP}(t), i_{wP}(t)) = \frac{\partial X}{\partial w} (t).$$

This approach is strictly true for infinitesimally small disturbances; however, experience has shown it to be valid in steady flight for disturbances of a sufficient magnitude to be of practical interest. Physically, the phrase "small disturbances" implies that if the stability derivatives (the coefficients in the Taylor series) are computed along a prescribed path and then a disturbed path is computed on the basis of the linearized equations, it is not possible to distinguish between the stability derivatives computed in the basis of the prescribed path plus the disturbed path compared to those computed along the prescribed path. Mathematically, we have expressed each side of equations V-12 as a zeroth-order term and then equated the zeroth-order terms to obtain the path equations, and the first-order terms to obtain the equations of the dynamics.

Practically speaking, we must decide whether any second-order terms would be significant for finite disturbances. Mathematically, the order of each term is clear; however, when the disturbances are of finite size, it becomes difficult to estimate precisely which higher-order terms will influence the solutions, and which will not. This would depend to a great extent on the detailed nature of the aerodynamic forces and considerations discussed at the beginning of this chapter.

B. Short Period Response

We now investigate in detail the nature of the response of an airplane in unsteady flight. The linearized equations of motion of an airplane with

no horizontal velocity perturbation ($\Delta u = 0$) from Section A and Appendix II, are:

$$\frac{dw}{dt} = Z_w w + U_P q, \quad (V-18a)$$

$$\frac{dq}{dt} = M_w w + M_q q. \quad (V-18b)$$

These equations describe what is usually referred to as the short period motion of the airplane, since they describe the initial, comparatively rapid motion of the aircraft. To the pilot, any serious deviations in the nature of the rapid motions of the aircraft would be of considerably more importance than effects on the slower motions, since accelerating or transition flight would be performed with the pilot's attentions firmly fixed on the task of controlling the aircraft.

Equations V-18 apply at constant velocity as well as when the flight velocity U_P , is changing. To simplify the notation, the subscript on U_P , the reference velocity, will be dropped. Equations V-18 are restricted to approximately level flight, otherwise a gravity term will be present in the vertical force equation. Control input terms are not included, since only the response to initial conditions is investigated.

The stability derivatives are, in general, a function of flight velocity, power setting δ_T , and wing tilt angle. These relationships are discussed in detail in Appendices II and III. For conventional aircraft as well as VTOL aircraft at low speeds, it is a reasonable approximation to consider the stability derivatives determining the short period characteristics as only functions of velocity. The term accounting for downwash lag ($M_{\dot{w}}$) usually included in airplane analyses (Reference 14) is neglected here. It is considered that the presence or absence of this derivative in these equations will not affect the general nature of the results.

The object of this analysis is to draw general conclusions regarding the influence of varying flight velocity on the transient response of an airplane. In particular, the discussion of Chapter III indicates that we have important effects of varying frequency on the amplitude of the short period motion. In addition, each variable of interest in the problem must be investigated, since each one, in general, will be a different function of time.

Smooth and monotonic variations of the stability derivatives with velocity are investigated. The major part of the study deals with

statically stable aircraft ($M_w < 0$). Some consideration is given to the case where M_w changes sign, as typical of VTOL aircraft at low speeds (Reference 24).

One objective of this study is to determine whether there are any unusual, or unforeseen phenomena that may occur due to variable velocity. Thus, consideration of unsteady effects on statically unstable aircraft, whose steady flight characteristics are poor, is not considered to be of particular importance at this time.

First we examine the variation of stability derivatives typical of conventional airplanes, where the stability derivatives are linearly proportional to flight velocity. Then we will investigate other variations, first with no airframe damping, and then with damping.

We define no airframe damping as:

$$\sigma_B \equiv 0 \equiv \left\{ \frac{L_w}{m} + \frac{D}{\mu} - \frac{M_q}{I} \right\} U. \quad (V-19)$$

This implies that the damping in pitch for a normal airplane is positive (unstable), since vertical damping is usually present ($\frac{L_w}{m} + \frac{D}{\mu} > 0$).

We consider the following cases:

1. Stability derivatives are linearly proportional to velocity.
- 2a. The attitude stability varies as a power of the velocity.
- 2b. The attitude stability varies linearly with velocity and is nonzero in hovering. This is a reasonable approximation to the variation of attitude stability with velocity for many VTOL aircraft.

1. Conventional Aircraft

We first study the short period response of a conventional airplane with the flight velocity varying in an arbitrary manner. The pitch damping (M_q), the attitude stability (M_w), and the slope of the lift curve (L_w) are assumed to be proportional to velocity. The drag is assumed to be proportional to velocity squared and the variation of drag with angle of attack is neglected.

Flight speeds down to and including zero are investigated. While the case of very low flight speeds is perhaps physically unrealistic, it will assist in understanding the effects of wide variations in unsteady flight conditions.

Very low flight speeds may be thought of as a rough approximation to a deflected jet VTOL, where induced flow effects from the engines are neglected.

The stability derivatives are:

$$\frac{M_q}{I} = C_{M_{\bar{q}}} \left[\frac{\rho S c}{2I} \right] U = C_{M_{\bar{q}}} \left[\frac{c}{2\mu k_y^2} \right] U \quad \bar{q} \equiv \frac{qc}{U}$$

$$\frac{M_w}{I} = C_{M_\alpha} \left[\frac{\rho S c}{2I} \right] U = C_{M_\alpha} \left[\frac{1}{2\mu k_y^2} \right] U$$

$$\frac{L_w}{m} = C_{L_\alpha} \left[\frac{\rho S}{2m} \right] U = C_{L_\alpha} \left[\frac{1}{2\mu c} \right] U \quad (V-20)$$

$$\frac{D}{m} = C_D \left[\frac{\rho S}{2m} \right] U^2 = C_D \left[\frac{1}{2\mu c} \right] U^2$$

where $C_{M_{\bar{q}}}$, C_{M_α} , C_{L_α} , and C_D are assumed to be independent of flight velocity. The relationships, V-20, are good approximations to the stability derivatives of an airplane at subsonic speeds without appreciable power effects (Reference 14).

The differential equation describing the pitching velocity, q , is (see Appendix II):

$$\begin{aligned} \frac{d^2 q}{dt^2} + \left[\frac{1}{2\mu c} \left(C_{L_\alpha} + C_D - C_{M_{\bar{q}}} \left(\frac{c}{k_y} \right)^2 \right) - \frac{\dot{U}}{U^2} \right] U \frac{dq}{dt} + \\ \frac{1}{4\mu^2 k_y^2} \left[-2\mu C_{M_\alpha} - C_{M_{\bar{q}}} (C_{L_\alpha} + C_D) \right] U^2 q = 0. \end{aligned} \quad (V-21)$$

The differential equation for the vertical velocity, w , is identical to V-21. Note that the fact that these two equations are identical is a special case, and is not generally true. This property of time-varying systems may be seen by observing that the differential equations for the angle of attack, α , and the normal acceleration, N_Z , are different from pitching velocity and vertical velocity:

$$\frac{d^2\alpha}{dt^2} + (2\sigma_B + \frac{\dot{U}}{U^2}) U \frac{d\alpha}{dt} + (\frac{U^2}{l_o^2} + \frac{\ddot{U}}{U} + [2\sigma_B - \frac{\dot{U}}{U^2}] \dot{U}) \alpha = 0, \quad (V-22)$$

$$\frac{d^2N_Z}{dt^2} + (2\sigma_B - \frac{3\dot{U}}{U^2}) U \frac{dN_Z}{dt} + (\frac{U^2}{l_o^2} - \frac{\ddot{U}}{U} - \dot{U} [2\sigma_B - 3 \frac{\dot{U}}{U^2}]) N_Z = 0, \quad (V-23)$$

where $\alpha \cong \frac{w}{U}$, $N_Z = [L_w + \frac{D}{U}] w$, and σ_B and l_o are defined by V-24.

In particular, the damping term differs due to the presence of the unsteady term $\frac{\dot{U}}{U}$ which will be reflected by different time histories of these two variables as compared to the pitching velocity.

We now nondimensionalize the velocity by $U^* = U_o \sqrt{1 + f_o}$ where

U_o is the initial flight velocity

f_o is the initial value of the parameter $f = \frac{\dot{U}}{\omega U}$

\dot{U}_o is the initial acceleration

ω_o is the initial short period frequency $= \frac{U_o}{l_o}$

Therefore,

$$U^* \rightarrow U_o \text{ as } f_o \rightarrow 0$$

$$U^* \rightarrow U_o \sqrt{f_o} \rightarrow \sqrt{\dot{U}_o l_o} \text{ as } f_o \rightarrow \infty.$$

Small values of f_o correspond to low accelerations, high initial velocity, and high frequencies, and large values correspond to low frequencies, slow initial velocities, and rapid accelerations.

We define

$$l_o = \frac{2\mu k_y^2}{\sqrt{-2\mu C_{M_\alpha} - C_{M_q}(C_{L_\alpha} + C_D)}}$$

$$2\sigma_B = \frac{1}{2\mu c} \left(C_{L_\alpha} + C_D - C_{M_q} \left(\frac{c}{k_y} \right)^2 \right) \quad (V-24)$$

where l_o is the wavelength of the short period motion. That is, it is the distance traveled by the airplane while executing one cycle of short period motion. This parameter is independent of flight speed and varies only with density for a given airplane.

The particular form of the characteristic velocity U^* is taken for two reasons:

1. U^* will never be zero except for the trivial case of hovering with no acceleration.
2. For small values of f , the effects of acceleration can be directly interpreted.

The time is nondimensionalized by

$$\tau = \frac{U^*}{l_o} t = \omega_o \sqrt{1+f_o} t. \quad (V-25)$$

This form of the characteristic time is selected for the same reasons as the selection of the characteristic velocity.

The differential equation for q in terms of these parameters is:

$$\frac{d^2 q}{d\tau^2} + \left[2\delta_B - \frac{\dot{U} l_o}{U^2} \right] \bar{U} \frac{dq}{d\tau} + \bar{U}^2 q = 0 \quad (V-26)$$

or

$$\frac{d^2 q}{d\tau^2} + \left[2\delta_B - \frac{\frac{d\bar{U}}{d\tau}}{\bar{U}^2} \right] \bar{U} \frac{dq}{d\tau} + \bar{U}^2 q = 0, \quad (V-27)$$

where δ_B , the steady flight short period damping ratio, is a constant. The characteristic time and velocity selected do not apply when $f_0 \leq -1$, corresponding to rapid decelerations. This, however, is an extreme value of f_0 and is not of particular significance for the following reason.

Since

$$f_0 \equiv \frac{\dot{U}_0}{\omega_0 U_0} = \frac{\dot{U}_0 P_0}{U_0 2\pi} = \frac{\Delta U_{\text{period}}}{2\pi U_0},$$

a value of $f_0 = -1$ indicates that in $\frac{1}{2\pi}$ of an initial period, the flight velocity is zero. The dynamic motion is only of interest for positive flight velocities. Therefore, when $f_0 \leq -1$, we may neglect the spring constant entirely, since the time interval of interest is only a very small part of a cycle. Thus, the range of interest of f_0 is $-1 < f_0 \leq \infty$.

To solve equation V-27, we transform the independent variable

$$\tau_1 = \int_0^{\tau} \bar{U} ds \quad (V-28)$$

where τ_1 may be considered as the nondimensional distance traveled. Then

$$\frac{dq}{d\tau} = \bar{U} \frac{dq}{d\tau_1} \quad \frac{d^2 q}{d\tau^2} = \frac{d^2 q}{d\tau_1^2} \bar{U}^2 + \frac{dq}{d\tau_1} \frac{d\bar{U}}{d\tau}.$$

The transformed differential equation describing q is:

$$\frac{d^2 q}{d\tau_1^2} + 2\delta_B \frac{dq}{d\tau_1} + q = 0. \quad (V-29)$$

Equation V-29 is a constant coefficient differential equation, and thus the differential equation V-21 is exactly integrable in terms of simple functions for any velocity time history. Note that equation V-29 is independent of the acceleration parameter, f_0 , and the time history of the velocity of the airplane. Variable flight velocity causes only a stretching or shrinking of the time scale, as seen from equation V-28.

The homogeneous solution to equation V-30 is:

$$q(\tau_1) = e^{-\delta_B \tau_1} \left(C_1 \cos \left[\sqrt{1 - \delta_B^2} \tau_1 + \Phi_1 \right] \right). \quad (V-30)$$

For simplicity, numerical results are presented only for constant acceleration and deceleration. However, the solution, V-30, is valid for any velocity variation with time.

To determine the pitching velocity response, we proceed in the following way. On a graph of the transformed independent variable, τ_1 , vs. nondimensional real time, τ , the relationship between τ_1 and τ given by equation V-28 is plotted for the velocity time history of interest. The response of the airplane as a function of τ_1 is plotted along the τ_1 axis. The response of the airplane in real time, proportional to τ , results from the particular relationship between τ_1 and τ for the given problem. Comparison of the actual τ_1 , τ curve to the line with unity slope indicates the distortion of the response in real time.

Figure 2 shows the relationship between τ and τ_1 for various values of f_0 with constant acceleration. Relationships are plotted both for $f_0 > 0$, representing accelerated flight, and $f_0 < 0$, representing decelerated flight. The curves for τ_1 vs. τ cease to have meaning when the slope becomes zero, since the slope $\frac{d\tau_1}{d\tau}$ is proportional to flight velocity. Note the small part of a cycle that is of interest when $f_0 \leq -.5$.

When f_0 is small, there is a range of τ over which little or no distortion occurs. Over this interval, the effects of nonsteady flight on the response may be approximated by a simple time scale change. The applicability of this simple approximation depends upon the magnitude of f_0 , as well as the time of observation. As f_0 increases, the time interval during which this approximation applies becomes smaller.

When the airplane is decelerating, the effects are more pronounced than when accelerating, as seen by comparing the departure of the curve for $f_0 = -.1$ from the $f_0 = 0$ line to that of $f_0 = .1$. The percentage change in velocity increases as the flight speed decreases.

The frequency of the motion changes with time, increasing if the airplane is accelerating and decreasing if the airplane is decelerating. When the airplane has no natural damping, there is no amplitude change; neutral stability exists for any value of f_0 . Typical responses are shown in Figure 3.

When the airplane decelerates to hover and stops, then the motion at zero velocity is $q = \text{constant}$, $w = \text{constant}$; i. e., there are no aerodynamic forces acting on the airplane, and it continues to move at a constant angular rate and vertical velocity. This simple phenomenon corresponds to missile motions called tumbling (Reference 9), and is of

little physical significance in this problem. This ultimate motion depends upon the initial conditions. For one set of initial conditions, the airplane would be precisely at rest at zero flight speed.

Because of the nondimensionalization used, the time scale of motion of an airplane in different flight conditions must be carefully interpreted. The relationship between flight speed, acceleration, the physical parameters of the airplane, and real time is given by V-25 as:

$$\tau = \frac{1}{l_0} \sqrt{U_0^2 + U_0 l_0} t.$$

For constant acceleration, the quantity under the square root sign is the velocity attained by the vehicle after traveling one-half a short period wavelength, $\frac{l_0}{2}$, and thus the characteristic time is the reciprocal of the instantaneous frequency of the short period motion after traveling one-half a short period wavelength. It is roughly an average period of the motion.

To determine the way in which the response of an airplane varies with initial flight velocity for the same acceleration, note that the time scale changes, as well as the time function describing the response (due to f_0). As the flight speed is reduced, for example, f_0 increases, resulting in a different transient motion in real time, shown by a change in the relationship between τ and τ_1 , and in addition, corresponding values of τ represent a longer real time interval when the same airplane (l_0 constant) and the same acceleration at each initial flight speed are studied. Figure 4 presents responses vs. a modified characteristic time

$\hat{\tau} = \sqrt{\frac{U_0}{l_0}}$, presenting directly the variation of the response with the time of occurrence of a disturbance as an airplane accelerates from hover to some flight speed.

The effects of airframe damping on the pitching velocity and vertical velocity responses will now be investigated. It can be observed from Figure 2, that depending upon the value of f_0 , the envelope of the response may be distorted either towards an apparent loss in damping as the τ_1 vs. τ curve lies below the $f_0 = 0$ line, or an apparent increase in damping as the τ_1 vs. τ curve lies above the $f_0 = 0$ line.

One way of presenting the magnitude of the damping variation is to compare the time to half amplitude of the response when the flight

velocity is varying to the time to half amplitude when the flight velocity is constant.

The percentage change, based on the steady flight time to half amplitude, is shown vs. $\frac{f_o}{1 + f_o}$ in Figure 5. A physical boundary occurs where the flight velocity is zero at the time to half amplitude.

The percentage variation is quite small, particularly when the steady flight damping ratio is of appreciable magnitude, and f_o is small. Significant changes occur when f_o is large and the damping ratio is either very small or very large. The former case is physically not particularly significant since the time to half amplitude is long. The effects at large damping ratios may be noticeable. The values of f_o necessary for large effects would indicate very low flight speeds.

Figure 6 shows a direct comparison of the envelope of the response for two damping ratios for the limiting values of $f_o = 0$, steady flight, and $f_o = \infty$. At small values of time, the response for $f_o = \infty$ appears somewhat less damped, and as time increases appears better damped, in comparison to steady flight. It should be recalled that the frequency of the motion is also changing, so that in the stretched time domain, what we might call the "instantaneous damping ratio" has not changed, since the frequency is different, as well as the amplitude for $f_o = \infty$ compared to $f_o = 0$. That is, if we imagine plotting a constant coefficient response on a rubber sheet, and then stretch or shrink the sheet, we change the apparent frequency and damping, but do not change the "instantaneous damping ratio."

This is a useful way to visualize the effects of varying flight velocity on the short period dynamics of an airplane. Recall that we have so far only investigated the response in pitching velocity and vertical velocity.

In general, the time histories of other variables may be of significance, and will be different due to the time-varying nature of the problem. In a piloted airplane, for example, the variation of normal acceleration with time is a quantity readily sensed by the pilot that may be of importance. Angle of attack or fuselage attitude may also be of interest. All of these variables will, in general, be described by functions of time that are different from the pitching velocity and the vertical velocity.

Thus, to obtain a complete analysis of the dynamic behavior of the airplane in nonsteady flight we must evaluate the response of all variables of interest. We will restrict detailed consideration of other variables to

the angle of attack and normal acceleration. The angle of attack response is determined by the relationship between angle of attack and vertical velocity;

$$\alpha = \tan^{-1} \frac{w}{U} \cong \frac{w}{U} ,$$

or directly from the differential equation V-22. The former approach is simpler. We assume that the arc tangent may be replaced by the angle in radians. While this approximation may not be valid at low speeds, it will preserve the trends in α . Errors incurred will not be of particular physical significance.

Then, the angle of attack response is

$$\alpha = \frac{1}{U} e^{-\delta_B \tau_1} \left[C_1 \cos \left(\sqrt{1 - \delta_B^2} \tau_1 + \Phi \right) \right] . \quad (V-31)$$

For constant acceleration, equation V-31 may be expressed as

$$\alpha = \frac{e^{-\delta_B \left[\frac{1+f_o}{2f_o} \right] \left(\left(\frac{1}{\sqrt{1+f_o}} + \frac{f_o}{1+f_o} \tau \right)^2 - \frac{1}{1+f_o} \right)}}{\sqrt{\frac{1}{1+f_o} + \frac{f_o}{1+f_o} \tau}} C_1 \cos \left(\sqrt{1 - \delta_B^2} \tau_1 + \Phi \right) . \quad (V-32)$$

When $f_o = \infty$ and $\tau = 0$, the flight velocity is zero, and the angle of attack is undefined.

The normal acceleration response is computed from the relationship between vertical velocity and normal acceleration:

$$N_Z = Z_w w ,$$

so

$$N_Z = \left[\frac{1}{2\mu c} \right] [C_{L_\alpha} + C_D] \bar{U} e^{-\delta_B \tau_1} C_1 \cos \left(\sqrt{1 - \delta_B^2} \tau_1 + \Phi \right) , \quad (V-33)$$

and for constant acceleration

$$N_Z = \left[\frac{1}{\sqrt{1+f_o}} + \frac{f_o}{1+f_o} \tau \right] e^{-\delta_B \left[\left(\frac{1}{\sqrt{1+f_o}} + \frac{f_o}{1+f_o} \tau \right)^2 - \frac{1}{1+f_o} \right]} C_1 \cos \left(\sqrt{1-\delta_B^2} \tau_1 + \Phi \right). \quad (V-34)$$

Inspection of equations V-31 and V-33 reveals that the amplitude of the angle of attack motion will decrease more rapidly in accelerating flight than the amplitude of pitching velocity, and not as rapidly in decelerating flight.

Normal acceleration exhibits the opposite tendency, being less stable in accelerating flight than the pitching velocity and more stable in decelerating flight.

Thus, while the influence of changing flight velocity on the pitching velocity and vertical velocity is to distort the time scale, the angle of attack and normal acceleration responses will have amplitude modifications as well, as anticipated by the difference in the differential equations for these variables (cf. equations V-21, V-22, and V-23).

Figure 7 is a contour map of the amplitude variation with time of the angle of attack and normal acceleration.

$\delta_B \left[\frac{1+f_o}{f_o} \right]$ vs. $\left[\frac{1}{1+f_o} + \frac{f_o}{1+f_o} \tau \right]$ is plotted. The former quantity depends upon the steady flight damping ratio, and the latter quantity is nondimensional velocity and is proportional to real time. Recall that $f_o > -1$.

$f_o > 0$ acceleration $f_o < 0$ deceleration.

$\delta_B > 0$ stable airplane in steady flight.

$\delta_B < 0$ unstable airplane in steady flight.

On Figure 7, as time increases, we move to the right for accelerating motion and to the left for decelerating motion.

We are located in the upper half of the graph when the airplane is stable and accelerating or is unstable and decelerating. We are located

in the lower half when the airplane is stable and decelerating or is unstable and accelerating.

In the upper half of the normal acceleration graph (stable-accelerating, unstable-decelerating) or in the lower half of the angle of attack graph (stable-decelerating, unstable-accelerating), we have a time history that during some time interval appears stable (unstable) and then appears unstable (stable).

That is, the stable, accelerating airplane (moving to right) may have a normal acceleration response that initially increases with time (the increasing Z_w dominating) and then decreases with time (the exponential damping term dominating), even though the airplane is stable in steady flight. Exactly the same phenomena may occur for the unstable, decelerating airplane.

The angle of attack response possesses opposite tendencies. A decelerating, stable airplane may exhibit a time history that initially decreases in amplitude to a minimum and then increases. Initially, the damping term dominates, and as time increases the velocity change dominates, the amplitude of the angle of attack response always approaching $\infty (90^\circ)$ as the airplane approaches hover.

Note that in the opposite regions of each graph, relating to the normal acceleration response for a stable, decelerating airplane, and the angle of attack response for a stable, accelerating airplane, the effects are quite orderly. The modifying amplitude factors and the airplane damping act in the same sense. Thus, in both of these situations, the normal acceleration in deceleration and the angle of attack during acceleration will be better damped than the pitching velocity and vertical velocity.

This, then, gives a reasonably complete description of the influence of changing flight velocity on the short period response of a conventional airplane.

For clarity, the response of the aircraft in the special case of $f_0 = \infty$ is shown separately in Figure 8. The angle of attack is initially undefined. Responses are shown for a damping ratio of 0 and .4. The angle of attack response is stable with no airframe damping, and the normal acceleration response is unstable. It would be expected that in low-speed flight the character of both of these curves is not particularly significant, since the flight behavior of the airplane is not strongly dependent upon angle of attack; i. e., the airplane is being lifted by jet thrust. Possible occurrence of wing stall would not be important due to the low dynamic

pressures. The lift curve slope is small, so that the levels of normal acceleration encountered are small.

To summarize these results, two features should be carefully noted.

1. There are compensating effects in the two-degree of freedom short period motions of an airplane, such that, for an airplane with no natural damping, the changing frequency with velocity (i. e., time) causes no amplitude change. This is somewhat different than might be expected from the discussion in Chapter III on time-varying equations, where an increasing frequency results in an apparent damping. This effect would be present in an airplane if the short period motion is approximated by a single-degree-of-freedom motion; i. e., if the center of gravity of the airplane is constrained to move in a straight line. For two-degree-of-freedom motion, the unbalance of the forces in the X direction enters into the equations, causing the resulting effects discussed.
2. The time histories of different variables are different functions of time. This is in contrast to the properties of constant coefficient systems where all variables are composed to the same functions of time, but is a natural consequence of multiple-degree-of-freedom systems with variable coefficients.

The unsteady effects might be thought of as being least on the pitching velocity and vertical velocity, where only the time scale is distorted. Somewhat stronger and unusual effects are noted in angle of attack and normal acceleration, where there is an additional envelope change.

The preceding method of presentation does not clearly show the effects of acceleration on the dynamics of an airplane at conventional flight speeds, so we present the identical results in an alternate form, which will not apply as the initial flight speed approaches zero ($f_0 \rightarrow \infty$).

The characteristic time is taken as the initial frequency, ω_0 , and the characteristic velocity is the initial flight velocity.

Figure 9 presents the relationship between $\bar{\tau}_1$, the transformed time, and $\bar{\tau}$, the nondimensional real time ($\omega_0 t$). Here the curves do not cross over the $f_0 = 0$ line (cf. Figure 2), so that accelerating flight results in a stretching of the time scale.

Time histories with no airframe damping for various values of f_0 are shown in Figure 10. The influence of different magnitudes of

acceleration and deceleration on the response of an airplane at a given initial flight speed are shown directly, since ω_0 is constant for a prescribed initial flight condition.

The frequency of the motion increases as velocity increases, and decreases as the velocity decreases.

Large changes in the angle of attack envelope are present, appearing as stable when $f_0 > 0$ and unstable when $f_0 < 0$. When $f_0 = -.2$, in fact, the angle of attack response diverges. The actual motions of the airplane are quite mild, as noted from the pitching velocity motion. However, it does not seem likely that this behavior of angle of attack is of real importance since a value of $\bar{\tau} = 4$ at $f_0 = -.2$ represents an instantaneous velocity of 20 percent of the initial velocity. For this divergence to occur rapidly then, the airplane initially would have been at a very low flight speed, where the angle of attack is not of particular importance.

The counterpart of Figure 5 is presented in Figure 11. The magnitude of the fractional change in time to one-half amplitude for pitching velocity and vertical velocity is presented as suggested in Reference 2. (However, as discussed in Chapter II, the analysis and graphs which are presented for angle of attack in Reference 2 are incorrect.) Figure 11 may be considered as indicative of under what flight conditions we may ignore the effects of acceleration or deceleration.

Figure 11 results from the following relationships. The percentage change in the time for the motion to damp to one-half amplitude is:

$$E = \frac{\bar{\tau}_{1\frac{1}{2}} - \bar{\tau}_{\frac{1}{2}}}{\bar{\tau}_{\frac{1}{2}}}, \quad (V-35)$$

where $\bar{\tau}_{\frac{1}{2}}$ is the time to one-half amplitude in steady flight (equal to .693 divided by the damping ratio) and $\bar{\tau}_{1\frac{1}{2}}$ is the time to one-half amplitude in accelerating or decelerating flight. For constant acceleration, the relationship between $\bar{\tau}_1$ and τ is given by

$$\bar{\tau}_1 = \bar{\tau} + \frac{f_0 \bar{\tau}^2}{2}. \quad (V-36)$$

Solving V-36 for $\bar{\tau}$ in terms of $\bar{\tau}_1$, the relationship between $\bar{\tau}_1$, f_0 , and E is as follows:

$$f_o \bar{\tau}_{\frac{1}{2}} = \frac{-2E}{(1 + E)^2}, \quad (V-37)$$

where E is negative in accelerating flight, ($f_o > 0$), indicating a shorter time to one-half amplitude, and positive in decelerating flight, ($f_o < 0$), indicating a longer time to one-half amplitude.

The relationship (V-37) is plotted in Figure 11. Note the asymmetry of the diagram which can also be clearly seen from Figure 9. The variation in time to half amplitude is greater in decelerating flight for the same magnitude of f_o , a natural consequence of the percentage velocity change increasing as the flight velocity decreases. Also, as before, a physical boundary exists for deceleration, since the time to half amplitude has no meaning when the flight velocity of the aircraft is negative at that time.

Recall that this discussion only applies to pitching velocity and vertical velocity.

We also evaluate the angle of attack response and the normal acceleration response as before.

Thus, for constant acceleration, in terms of the parameters used here,

$$\alpha = \frac{w(t)}{U_o (1 + f_o \bar{\tau})}, \quad (V-38)$$

and therefore the envelope of the angle of attack response may be expressed as

$$(\alpha)_e \sim \frac{e^{-\frac{\delta_B}{f_o} \left[f_o \bar{\tau} + \frac{(f_o \bar{\tau})^2}{2} \right]}}{(1 + f_o \bar{\tau})}, \quad (V-39)$$

where δ_B is the damping ratio of the airplane response in steady flight.

$\bar{\tau}_{\frac{1}{2}}$ is the time (nondimensional) to one-half amplitude in steady flight.

Now, in terms of this parameter we may express the angle of attack envelope as

$$(\alpha)_e \sim e^{-\frac{.693}{f_0 \bar{\tau}_\frac{1}{2}} \left[f_0 \bar{\tau} + \frac{(f_0 \bar{\tau})^2}{2} \right] - \ln(1 + f_0 \bar{\tau})} \quad (V-40)$$

For convenience, define $\eta_1 = f_0 \bar{\tau}_\frac{1}{2}$, $\eta = f_0 \bar{\tau}$, and the relationship between the amplitude ratio AR, η and η_1 is:

$$\eta_1 = \frac{.693 \left(\eta + \frac{\eta^2}{2} \right)}{\ln(AR(1 + \eta))} \quad (V-41)$$

Also for constant acceleration, using the relationship between vertical velocity and normal acceleration, $N_Z = Z_w w$, the envelope is

$$(N_Z)_e \sim (1 + f_0 \bar{\tau}) e^{-\delta_B \left(\bar{\tau} + \frac{f_0 \bar{\tau}^2}{2} \right)}; \quad (V-42)$$

we may develop a similar relationship between the parameters η_1 and η and the amplitude ratio of the response. The result is

$$\eta_1 N_Z = \frac{.693 \left(\eta + \frac{\eta^2}{2} \right)}{\ln \left(\frac{\eta + 1}{AR} \right)} \quad (V-43)$$

Figure 12 presents contour maps of the amplitude of the angle of attack and the normal acceleration based on equations V-41 and V-43. The vertical axis ($\eta_1 = f_0 \bar{\tau}_\frac{1}{2}$) depends upon the acceleration parameter and the time to one-half amplitude in steady flight. The horizontal axis ($\eta = f_0 \bar{\tau}$) is proportional to real time, the time scale being set by the acceleration parameter f_0 . Since $\bar{\tau} > 0$, we move to the right when the airplane is accelerating, and to the left when the airplane is decelerating. When the airplane is stable in steady flight $\bar{\tau}_\frac{1}{2} > 0$, and if it is unstable $\bar{\tau}_\frac{1}{2} < 0$.

Thus the steady flight stability and whether the airplane is accelerating or decelerating locates us in different quadrants:

first quadrant - stable, accelerating
 second quadrant - unstable, decelerating
 third quadrant - stable, decelerating
 fourth quadrant - unstable, accelerating

Figure 12 presents the same information as Figure 7 in a different form. Since Figure 7 has been considered in some detail, we will not discuss Figure 12 in detail. We note that the unusual envelope behavior occurs in the first and second quadrants for normal acceleration. $f \bar{\tau}_{\frac{1}{2}} > .693$ before the apparent instability in normal acceleration appears.

For angle of attack, the unusual envelope behavior occurs in the third and fourth quadrants. We note here that if $f \bar{\tau}_{\frac{1}{2}} < -.693$, the angle of attack response will always appear unstable in unsteady flight. The significance of these critical points is interpreted by recalling that

$$\bar{\tau}_{\frac{1}{2}} = \frac{.693}{\delta_B},$$

and thus normal acceleration will exhibit an instability over some time interval in accelerating flight when

$$f_o > \delta_B. \quad (V-44)$$

Angle of attack will always be unstable when

$$f_o < -\delta_B \quad (V-45)$$

for stable aircraft. Note that V-45 is not a stability criterion for angle of attack, since it is always ultimately unstable when f_o is negative and the acceleration is constant. Relationships V-44 and V-45 give an indication of a way of estimating the significance of the size of f_o that will be discussed in more detail in the following pages.

Some typical angle of attack and normal acceleration envelopes illustrating the above phenomena are shown in Figure 12. We again clearly see the additional complication of dealing with time variable systems, the fact that the envelopes of the various responses behave differently. Stability in an abstract sense is not particularly meaningful, and we must consider the particular variables that are of concern in the problem and determine stability criteria for them.

For example, in this case it is probably not of much importance that for the stable airplane, decelerating, the angle of attack response is essentially unstable. Unless an airplane is capable of very large decelerations, this would only occur at low flight speeds, and should not be of much concern to the pilot. However, it may be of some importance that in accelerating flight for the stable airplane the normal acceleration may appear initially unstable, although it is ultimately stable

We must, therefore, in investigating the dynamics of aircraft or missiles in nonsteady flight, consider carefully which variables are of significance in piloting the airplane, or from structural or other considerations, and study the nature of these variables. In many of the missile analyses, the distinction between angle of attack and normal acceleration has not been noted.

This, then, gives a reasonably complete picture of the initial condition response of the airplane for constant acceleration. Variable acceleration will affect the details of the results and may be easily evaluated in any specific case. Since the results are a function of the instantaneous value of the velocity, constant acceleration and deceleration should point out the significant effects.

Recall that other variables not discussed will have different time histories which may be of importance in specific situations. For example, airplane attitude, $\int_0^t q \, ds$, or angular acceleration $(\frac{dq}{dt})$ will be different from q .

For completeness, we consider a practical situation, resulting in a variable acceleration time history. This is the following problem. Consider an airplane in which the power is suddenly cut off; the airplane decelerates due to the drag force. This example is equivalent to the missile analyses with constant density, and is interesting since it is apparently one of the few physical problems which gives rise to Euler's differential equation.

The equation describing the trajectory of the airplane (i. e., the equation determining the velocity time history) is: (It is assumed that lift is maintained equal to the weight of the airplane as speed decreases.)

$$\begin{aligned} \frac{D_o}{m U_o} \left(\frac{U}{U_o} \right)^2 &= - \frac{\dot{U}}{U_o} \\ f_o \omega_o (\bar{U})^2 &= \dot{\bar{U}} \\ \frac{D_o}{m U_o} &= - f_o \omega_o \end{aligned} \tag{V-46}$$

The solution to equation V-46 is:

$$\bar{U} = \frac{1}{1 - f_o \bar{\tau}} \quad (V-47)$$

Defining σ_B and σ_w as:

$$2\sigma_B = \frac{L_{w_o}}{m} - \frac{M_{q_o}}{I} + \frac{D_o}{m U_o} \quad (V-48)$$

$$2\sigma_B = 2\sigma_w - f_o \omega_o$$

Here we note carefully the manner in which the drag enters into the damping term. To be consistent, the relationship between the drag and the acceleration parameter f_o must be considered. These quantities are related by the trajectory equation. When flight speed is varied by other factors, then the simple relationship (V-46) between drag and the acceleration parameter does not exist.

Since
$$f = \frac{U_o \dot{U}}{\omega_o U^2} = f_o$$

f is a constant. In other words, the fractional change in the spring constant per cycle is constant, a property of Euler's equation mentioned in Chapter III.

The differential equation for pitching velocity, as given previously, is:

$$\frac{d^2 q}{d\bar{\tau}^2} + (2\sigma_B - f_o) \bar{U} \frac{dq}{d\bar{\tau}} + \bar{U}^2 q = 0 \quad (V-26)$$

We solve equation V-26, where \bar{U} is given by V-47, by assuming a solution of the form

$$q = C (1 - f_o \bar{\tau})^N \quad (V-49)$$

Substituting V-49 into equation V-26 gives an equation for N :

$$N^2 - \frac{2\sigma_B}{f_o} N + \frac{1}{f_o^2} = 0 \quad (V-50)$$

When the two values of N obtained from V-50 are distinct, the two linearly independent solutions of equation V-26 are:

$$\left. \begin{matrix} q \\ w \end{matrix} \right\} = (1 - f_o \bar{\tau}) \left[\begin{matrix} \frac{\delta_B}{f_o} \left[\frac{i}{f_o} \sqrt{1 - \delta_B^2} & - \frac{1}{f_o} \sqrt{1 - \delta_B^2} \right] \\ C_1 (1 - f_o \bar{\tau}) & + C_2 (1 - f_o \bar{\tau}) \end{matrix} \right] \quad (V-51)$$

where it is assumed that $N_{1,2}$ are complex.

Taking the limit as $f_o \rightarrow 0$, the deceleration approaches zero and

$$\left. \begin{matrix} q \\ w \end{matrix} \right\} \rightarrow e^{-\delta_B \bar{\tau}} \left[\begin{matrix} C_1 \cos \sqrt{1 - \delta_B^2} \bar{\tau} + C_2 \sin \sqrt{1 - \delta_B^2} \bar{\tau} \end{matrix} \right], \quad (V-52)$$

and we recover the conventional steady flight result.

From the relationships between α and N_Z and w , the envelopes of the variables are:

$$\begin{aligned} (q)_e, (w)_e &\sim (1 - f_o \bar{\tau})^{\frac{\delta_B}{f_o}} = (1 - f_o \bar{\tau})^{\frac{\delta_w}{f_o} - \frac{1}{2}} \\ (\alpha)_e, (\theta)_e &\sim (1 - f_o \bar{\tau})^{1 + \frac{\delta_B}{f_o}} = (1 - f_o \bar{\tau})^{\frac{\delta_w}{f_o} + \frac{1}{2}} \\ (N_Z)_e &\sim (1 - f_o \bar{\tau})^{-1 + \frac{\delta_B}{f_o}} = (1 - f_o \bar{\tau})^{\frac{\delta_w}{f_o} - \frac{3}{2}}. \end{aligned} \quad (V-53)$$

The second form should be used to evaluate the effects of f_o , since δ_w is not dependent upon f_o , while δ_B is.

The parameter that determines the importance of deceleration is:

$$\frac{\delta_w}{f_o} = \left(\frac{L}{D}\right) \frac{1}{(\alpha - \alpha_{oL})} - \frac{C_{m\bar{q}_o}}{C_{D_o}} \left(\frac{c}{k_y}\right)^2. \quad (V-54)$$

For conventional aircraft with angular damping, this will be quite a large number. For typical values of this parameter, the first-order effects of deceleration due to drag may be obtained by expanding the envelope expressions in series in $f_o \bar{\tau}$, retaining only the first-order term:

$$\begin{aligned} (q)_e &\sim (1 - f_o \bar{\tau}) - \frac{1}{2} + \frac{\delta_w}{f_o \rightarrow e} - (\delta_w - \frac{f_o}{2}) \bar{\tau} \\ (\alpha)_e &\sim (1 - f_o \bar{\tau}) \frac{1}{2} + \frac{\delta_w}{f_o \rightarrow e} - (\delta_w + \frac{f_o}{2}) \bar{\tau} \end{aligned} \quad (V-55)$$

$$(N_Z)_e \sim (1 - f_o \bar{\tau}) - \frac{3}{2} + \frac{\delta_w}{f_o \rightarrow e} - (\delta_w - \frac{3}{2} f_o) \bar{\tau}$$

for small $f_o \bar{\tau}$.

These first-order approximations show clearly the differences in the envelopes of the motion, as well as the influence of the varying spring on damping. In addition, the difference between the quasi-steady solution and the true unsteady solution is clearly shown. A quasi-steady solution of equation V-26 with the relationship V-48 gives a damping term that is to the first order:

$$- 2 (\delta_w - f_o) \bar{\tau}.$$

From the quasi-steady solution, we would estimate the envelope of pitching velocity as $e^{-(\delta_w - f_o) \bar{\tau}}$. Comparison of the quasi-steady expression with the exact solution (V-55) shows that the varying spring constant has decreased the apparent damping by $\frac{f_o}{2}$. A similar difference is present in the angle of attack and normal acceleration envelopes. (Compare the equations V-55 with the damping terms in equations V-22 and V-23.) This points out the care with which we must interpret quasi-steady effects;

i. e., effects seen by inspection of the coefficients only. The true damping is less than the quasi-steady damping ($f_o < 0$), due to the decreasing spring constant.

Now let us turn to the question of how important these effects are. From V-46,

$$f_o = - \frac{D_o}{m U_o \omega_o} = - \frac{g}{U_o} \left(\frac{D}{L} \right)_o \frac{1}{\omega_o} .$$

For a conventional airplane, $f_o \geq -.1$. The importance of the unsteady effects are shown by comparing f_o to the damping ratio (see equations V-53), thus indicating that changes in the response due to deceleration are negligible except for an airplane with very low damping in steady flight.

Figure 13 presents response envelopes for very light airframe damping. For normal aircraft, with a typical lift curve slope, these cases would correspond to unstable damping in pitch.

Thus, it appears that there are only small changes in the dynamic stability characteristics of aircraft due to deceleration by drag, except in cases where the steady flight characteristics are already poor.

The results in Reference 2 on the short period motion in level flight are inconsistent in relation to the importance of the various parameters in this problem. The direct appearance of f_o in the damping coefficient of the angle of attack equation is neglected in this reference. From the foregoing analysis, however, it can be seen that the influence of the varying frequency on the amplitude motion is the same order of magnitude as this term. Therefore, the results of Reference 2 are essentially incorrect. As mentioned, the resulting effects are not large, so the neglect of the f_o term in the damping for a conventional airplane is perhaps reasonable, but inconsistent.

Note that in contrast to the constant acceleration case, the rate of change of the angle of attack envelope with time is always of uniform sign. Here the airplane only reaches zero velocity as time approaches infinity.

a. Comparison of Results with Approximate Approach on Complex Plane

To conclude the investigation of the short period motion of a conventional airplane, we demonstrate the approximate approach developed in

Chapter IV. The differential equations for pitching velocity and normal acceleration are:

$$\frac{d^2 q}{d\bar{\tau}^2} + [2\delta_B - f(\bar{\tau})] \bar{U} \frac{dq}{d\bar{\tau}} + \bar{U}^2 q = 0. \quad (V-21)$$

$$\frac{d^2 N_Z}{d\bar{\tau}^2} + [2\delta_B - 3f(\bar{\tau})] \bar{U} \frac{dN_Z}{d\bar{\tau}} + \bar{U}^2 [1 - f(\bar{\tau})(2\delta_B - 3f(\bar{\tau}))] N_Z = 0. \quad (V-23)$$

For simplicity, the term involving \ddot{U} in equation V-23 has been neglected.

We wish to interpret the short period results obtained in the previous section on the complex plane in accordance with the discussion of Chapter IV. We proceed in three steps:

1. Plot the frozen system locus. This is the locus of roots that would be obtained from a conventional point-by-point stability analysis, and would be identical for all variables. ($f(\bar{\tau}) \equiv 0$ in equations V-21 and V-23.)
2. Plot the quasi-steady locus. This is the locus of roots, taking into account any new terms that appear in the coefficients of the differential equations arising when the equations for each variable are determined. This locus will be different, in general, for each variable. For pitching velocity, the quasi-steady locus differs from the frozen locus due to the appearance of $[-f(\bar{\tau})]$ in the damping term. The terms $\{-3f(\bar{\tau})\}$ and $\{-f(\bar{\tau})[2\delta_B - 3f(\bar{\tau})]\}$ are included in the quasi-steady locus for normal acceleration. Recall that we have a system in which the frequency of motion is primarily determined by the coupling terms, so the quasi-steady solution does not necessarily reflect an improvement over the frozen system.
3. Plot the unsteady locus. This is the modification to the quasi-steady locus that takes into account the effects of the varying coefficients on the nature of the solutions to the differential equation. The root locus technique of Chapter IV is used.

In detail then, let us follow through a sample case for both normal acceleration and pitching velocity.

The frozen system equations are from V-21 and V-23, with $f(\bar{\tau}) \equiv 0$.

$$\frac{d^2 q}{d\bar{\tau}^2} + 2\delta_B \bar{U} \frac{dq}{d\bar{\tau}} + \bar{U}^2 q = 0, \quad (V-56)$$

$$\frac{d^2 N_Z}{d\bar{\tau}^2} + 2\delta_B \bar{U} \frac{dN_Z}{d\bar{\tau}} + \bar{U}^2 N_Z = 0, \quad (V-57)$$

and the frozen roots are

$$\lambda_{f1,2} = -\bar{U} \left\{ \delta_B \pm \sqrt{1 - \delta_B^2} i \right\}, \quad (V-58)$$

where the damping ratio is taken to be less than critical. This locus is shown in Figure 14, and represents the steady flight dynamics of the airplane at various speeds.

The unsteady system equations are V-21 and V-23:

$$\frac{d^2 q}{d\bar{\tau}^2} + [2\delta_B - f(\bar{\tau})] \bar{U} \frac{dq}{d\bar{\tau}} + \bar{U}^2 q = 0 \quad (V-21)$$

and

$$\frac{d^2 N_Z}{d\bar{\tau}^2} + [2\delta_B - 3f(\bar{\tau})] \bar{U} \frac{dN_Z}{d\bar{\tau}} + \bar{U}^2 [1 - f(\bar{\tau}) (2\delta_B - 3f(\bar{\tau}))] N_Z = 0. \quad (V-23)$$

The quasi-steady locus is obtained by considering equations V-21 and V-23 as though they were constant coefficient equations and solving for the "characteristic roots." Note that the quasi-steady roots for pitching velocity are different from the quasi-steady roots for normal acceleration; i. e.,

$$r_{q1,2} = -\bar{U} \left[\left(\delta_B - \frac{f(\bar{\tau})}{2} \right) \pm \sqrt{1 - \left(\delta_B - \frac{f(\bar{\tau})}{2} \right)^2} i \right], \quad (V-59)$$

$$r_{N_{Z1,2}} = -\bar{U} \left[\left(\delta_B - \frac{3}{2} f(\bar{\tau}) \right) \pm \sqrt{1 - f(\bar{\tau}) (2\delta_B - 3f(\bar{\tau})) - \left(\delta_B - \frac{3}{2} f(\bar{\tau}) \right)^2} i \right]. \quad (V-60)$$

Now to evaluate the actual unsteady effects we use the root locus technique developed in Chapter IV. In a second-order system, this step could be combined with the previous one. Algebraically, this combined procedure becomes complex when higher-order systems are examined. In addition, a step-by-step procedure makes the source of the various effects clear. In many physical problems, it may be possible that either the quasi-steady or the unsteady effects are not important. Usually, though, it appears from the preceding discussions that both effects will be of a similar magnitude, particularly with regard to the specific problem we are studying. Here, the quasi-steady effects on pitching velocity are actually in the wrong direction and are canceled by the unsteady effects.

Now, we use the quasi-steady roots as the base for the root locus including unsteady effects.

The unsteady "characteristic equation" is (see IV-12):

$$(\lambda_1 - r_1)(\lambda_1 - r_2) + r_1 = 0, \quad (V-61)$$

where the quasi-steady roots referring to a particular variable, r_1 , will be different for different roots (r_2 will be the conjugate of r_1 , for a complex pair). r_1 is the "gain," and is in general, complex, so the root locus drawn to evaluate the unsteady effects will not be the usual 180° or 0° locus but will be some other angle condition.

These loci are also shown in Figure 14.

These approximate results agree with the exact results. It should be recalled that this approach is not valid as the fractional change in frequency per cycle becomes large; however, it does seem to give the proper indications, even as the approximation breaks down.

The effects of various flight velocity time histories are visualized directly from this approach, since any variation of $f(\bar{T})$ can be easily included. For the unsteady locus, if the roots are only a function of velocity, then

$$\frac{dr}{dt} = \frac{dr}{dU} \frac{dU}{dt}, \quad (V-62)$$

and therefore the root locus gain is proportional to $\frac{dU}{dt}$.

This example shows again the difference between the stability of the pitching velocity response and the normal acceleration response. The

particular parameters selected indicate a normal acceleration response that initially appears unstable. The loci are shown only for increasing velocity.

To conclude the discussion, the response of a conventional airplane in unsteady flight, in various variables, in the short period mode, may be written as

$$q = e^{-\delta_B \int_0^\tau \bar{U} ds} C_1 \cos \left\{ \sqrt{1 - \delta_B^2} \int_0^\tau \bar{U} ds + \Phi_1 \right\}$$

$$w = e^{-\delta_B \int_0^\tau \bar{U} ds} C_2 \cos \left\{ \sqrt{1 - \delta_B^2} \int_0^\tau \bar{U} ds + \Phi_2 \right\}$$

$$\alpha = \frac{e}{\bar{U}(\tau)} e^{-\delta_B \int_0^\tau \bar{U} ds} C_3 \cos \left\{ \sqrt{1 - \delta_B^2} \int_0^\tau \bar{U} ds + \Phi_3 \right\}$$

$$N_Z = \bar{U}(\tau) e^{-\delta_B \int_0^\tau \bar{U} ds} C_4 \cos \left\{ \sqrt{1 - \delta_B^2} \int_0^\tau \bar{U} ds + \Phi_4 \right\}.$$

These equations are exact and apply for any velocity time history.

Stability criteria may be obtained by applying the condition that the amplitude of the envelope of the motion always decreases with time.

$$\text{For } q, w \quad \delta_B(t) > 0$$

$$\alpha \quad \delta_B(t) + f(t) > 0$$

$$N_Z \quad \delta_B(t) - f(t) > 0$$

Thus the importance of the unsteady effects on the stability is indicated by the magnitude of $f(t)$ compared to the instantaneous damping ratio $\delta_B(t)$.

2. Unconventional Stability Derivative Variations

a. Power Law Variation of Attitude Stability With Velocity

Now we investigate the importance of other variations of the attitude stability with velocity. We will assume that the attitude stability varies as a power of the flight velocity. Initially, we consider an airplane with no damping. For the relationship between velocity and attitude stability we take:

$$-M_w = C_o^a U^N. \quad (V-63)$$

Note that when $N < 0$, this form does not apply in hovering.

The differential equations for q and w are:

$$\frac{d^2 w}{dt^2} - \frac{\dot{U}}{U} \frac{dw}{dt} + C_o^a U^{N+1} w = 0, \quad (V-64)$$

$$\frac{d^2 q}{dt^2} - \frac{N\dot{U}}{U} \frac{dq}{dt} + C_o^a U^{N+1} q = 0. \quad (V-65)$$

The differential equations for q and w are now different, except when $N = 1$, the case investigated in Section 1.

With the nondimensionalization used previously, the characteristic velocity and time are taken as:

$$U^* = U_o (1 + f_o)^{\frac{2}{N+3}} \quad (V-66)$$

$$\tau = \omega_o (1 + f_o)^{\frac{N+1}{N+3}} t = C_o U^*^{\frac{N+1}{2}} t.$$

The differential equations V-64 and V-65 in nondimensional form are:

$$\frac{d^2 q}{d\tau^2} - \frac{N}{\bar{U}} \frac{d\bar{U}}{d\tau} \frac{dq}{d\tau} + (\bar{U})^{N+1} q = 0, \quad (V-67)$$

$$\frac{d^2 w}{d\tau^2} - \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau} \frac{dw}{d\tau} + (\bar{U})^{N+1} w = 0. \quad (V-68)$$

And now we transform the time by

$$\tau_1 = \int_0^{\tau} \bar{U}^{\frac{N+1}{2}} ds, \quad (V-69)$$

to obtain

$$\frac{d^2 q}{d\tau_1^2} + \left(\frac{1-N}{2}\right) \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1} \frac{dq}{d\tau_1} + q = 0, \quad (V-70)$$

$$\frac{d^2 w}{d\tau_1^2} + \left(\frac{N-1}{2}\right) \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1} \frac{dw}{d\tau_1} + w = 0. \quad (V-71)$$

These are the differential equations in a distorted time scale. When $N = 1$, the middle term is zero, and equations V-70 and V-71 are constant coefficient equations. When $N \neq 1$, these are variable coefficient equations, and we have additional unsteady effects. These equations may be written in terms of the acceleration parameter f .

$$f = \frac{\dot{U}}{\omega U} = \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1}.$$

$$\frac{d^2 q}{d\tau_1^2} + \left(\frac{1-N}{2}\right) f(\tau_1) \frac{dq}{d\tau_1} + q = 0, \quad (V-70a)$$

$$\frac{d^2 w}{d\tau_1^2} + \left(\frac{N-1}{2}\right) f(\tau_1) \frac{dw}{d\tau_1} + w = 0. \quad (V-71a)$$

We note that, when $f(\tau_1)$ is a constant, these equations are constant coefficient equations. It can be shown that this implies that the original equation is Euler's differential equation.

We now may proceed in two ways. The first is to assume a velocity-time relationship and then try to solve V-70 and V-71 exactly. Or we may use the asymptotic solutions, in which case the solution can be expressed directly without consideration of the velocity-time relationship.

First we consider exact solutions for constant acceleration, and then show the correspondence to the second approach when the asymptotic solution applies and generalize our result.

With acceleration constant, the instantaneous velocity is:

$$U(t) = U_o + U_o t$$

$$\bar{U}(t) = \frac{1}{(1 + f_o)^{\frac{N+3}{2}}} + \frac{f_o}{1 + f_o} \tau \quad (V-72)$$

$$\frac{dU}{d\tau_1} \frac{d\tau_1}{d\tau} = \frac{f_o}{1 + f_o}$$

and

$$\frac{d\tau_1}{d\tau} = \bar{U}^{\frac{N+1}{2}}, \quad (V-73)$$

and so

$$\frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1} = \frac{1}{\bar{U}^{\frac{N+3}{2}}} \left(\frac{f_o}{1 + f_o} \right). \quad (V-74)$$

So the differential equations V-72a and V-70b become:

$$\frac{d^2 q}{d\tau_1^2} + \left\{ \frac{1 - N}{3 + N} \right\} \left\{ \frac{1}{\tau_1 + \frac{1}{f_o} \left(\frac{2}{N+3} \right)} \right\} \frac{dq}{d\tau_1} + q = 0, \quad (V-75)$$

$$\frac{d^2 w}{d\tau_1^2} + \left\{ \frac{N - 1}{N + 3} \right\} \left\{ \frac{1}{\tau_1 + \frac{1}{f_o} \left(\frac{2}{N+3} \right)} \right\} \frac{dw}{d\tau_1} + w = 0. \quad (V-76)$$

If $f_o = 0$, corresponding to steady flight, the middle term is zero. In the special case $N = -3$, these are constant coefficient equations and the original equation was Euler's equation.

Equations V-75 and V-76 are a form of Bessel's equation ($N = -3$). The equations are placed in standard form by the following substitutions.

Let

$$\tau_2 = \tau_1 + \frac{1}{f_o} \left(\frac{2}{N+3} \right), \quad (V-77)$$

and transform the dependent variables by

$$q = Q e^{-\frac{1}{2} \left(\frac{1-N}{3+N} \right) \int^{\tau_2} \frac{ds}{s}}, \quad (V-78)$$

$$w = W e^{-\frac{1}{2} \left(\frac{N-1}{N+3} \right) \int^{\tau_2} \frac{ds}{s}}. \quad (V-79)$$

Equations V-75 and V-76 become:

$$\frac{d^2 Q}{d\tau_2^2} + \left\{ 1 + \frac{(1-N)(5+3N)}{4(N+3)^2} \frac{1}{\tau_2^2} \right\} Q = 0, \quad (V-80)$$

$$\frac{d^2 W}{d\tau_2^2} + \left\{ 1 + \frac{(N-1)(N+7)}{4(N+3)^2} \frac{1}{\tau_2^2} \right\} W = 0. \quad (V-81)$$

From Reference 18, page 146, we see that the solutions to the differential equations V-80 and V-81 are:

$$Q = \sqrt{\tau_2} \left\{ C_1 J_{\left(\frac{N+1}{N+3} \right)}(\tau_2) + C_2 J_{-\left(\frac{N+1}{N+3} \right)}(\tau_2) \right\}, \quad (V-82)$$

$$W = \sqrt{\tau_2} \left\{ C_1 J_{\left(\frac{2}{N+3} \right)}(\tau_2) + C_2 J_{-\left(\frac{2}{N+3} \right)}(\tau_2) \right\}, \quad (V-83)$$

where J_p is a Bessel function of the first kind of order p . The form given is valid when $\frac{N+1}{N+3}$ and $\frac{2}{N+3}$ are not integers (Reference 18).

Thus, whenever the attitude stability varies as a power of the velocity, and the acceleration is constant, we can express the response in terms of Bessel functions.

In terms of aircraft variables, equations V-82 and V-83 become:

$$q = (\tau_2)^{\frac{1+N}{3+N}} \left\{ Z_{\frac{1+N}{3+N}}(\tau_2) \right\}, \quad (V-84)$$

$$w = (\tau_2)^{\frac{2}{N+3}} \left\{ Z_{\frac{2}{3+N}}(\tau_2) \right\}, \quad (V-85)$$

where $Z_p = C_1 J_p + C_2 J_{-p}$.

The solutions are well behaved for all values of τ_2 . As $\tau_2 \rightarrow 0$, Z_p can be expressed in a series (Reference 18, page 128):

$$q \cong C_1' + C_2' \tau_2^{2(\frac{1+N}{3+N})}, \quad (V-86)$$

$$w \cong C_3' + C_4' \tau_2^{2(\frac{2}{3+N})}. \quad (V-87)$$

For large values of τ_2 , Z_p can be expressed in terms of asymptotic expansions (Reference 18, page 138):

$$q \cong (\tau_2)^{\frac{1}{2}(\frac{1+N}{3+N})} \{ C_1'' \cos(\tau_2 + \Phi_1) \}, \quad (V-88)$$

$$w \cong (\tau_2)^{\frac{1}{2}(\frac{1-N}{3+N})} \{ C_3'' \cos(\tau_2 + \Phi_2) \}. \quad (V-89)$$

For large values of τ_2 , the asymptotic expansion is a good approximation. What we mean by a large value depends upon the order of the Bessel function, as well as the value of τ_2 . If the order is reasonably small ($p < 1$), when $\tau_2 \geq 2$, V-88 and V-89 are reasonably accurate representations of the solution.

There is a smooth connection between the expressions for small τ_2 and large τ_2 , but it is difficult to express this connection analytically. The range of validity of V-86 and V-87 can be increased by taking more terms of the series, and the asymptotic expansions can be extended by using asymptotic series (Reference 18), however we will not consider these refinements here.

τ_2 is related to the physical parameters of the problem by equation V-77:

$$\tau_2 = \tau_1 + \frac{2}{N+3} \frac{1}{f_0} . \quad (V-77)$$

The initial value of f , f_0 , determines whether the asymptotic expansion applies over the time interval of interest. As $f_0 \rightarrow \infty$, the asymptotic expansion is not valid for small τ_1 , but will apply as τ_1 increases.

From V-77 and V-72,

$$\tau_2 = \left\{ \frac{1+f_0}{f_0} \right\} \left\{ \frac{2}{N+3} \right\} \bar{U}^{\frac{N+3}{2}} , \quad (V-90)$$

so we can express V-84 and V-85 in terms of f_0 and the velocity:

$$q = \bar{U}^{\frac{1+N}{2}} \left\{ Z \frac{1+N}{3+N} \left[\frac{1+f_0}{f_0} \cdot \frac{2}{N+3} \bar{U}^{\frac{N+3}{2}} \right] \right\} , \quad (V-91)$$

$$w = \bar{U} \left\{ Z \frac{2}{3+N} \left[\frac{1+f_0}{f_0} \cdot \frac{2}{N+3} \bar{U}^{\frac{N+3}{2}} \right] \right\} . \quad (V-92)$$

The general nature of the response can be simply interpreted when asymptotic expansions are valid. In this case, equations V-91 and V-92 become:

$$q \rightarrow (\bar{U})^{\frac{N-1}{4}} C_1'' \cos \left[\frac{1+f_0}{f_0} \frac{2}{N+3} \bar{U}^{\frac{N+3}{2}} + \Phi_1'' \right] , \quad (V-93)$$

$$w \rightarrow (\bar{U})^{\frac{1-N}{4}} C_2'' \cos \left[\frac{1+f_0}{f_0} \frac{2}{N+3} \bar{U}^{\frac{N+3}{2}} + \Phi_2'' \right] , \quad (V-94)$$

where \bar{U} is given by equation V-72.

The amplitude of pitching velocity and vertical velocity are different, and depend upon the power law. For accelerating flight, \bar{U} increasing with time, if $N > 1$, the amplitude of pitching velocity increases with time and the amplitude of vertical velocity decreases with time. The reverse is true in deceleration. If $N < 1$, the pitching velocity amplitude will increase in accelerating flight and the vertical velocity amplitude will decrease, when no airframe damping is present. The envelope growth is dependent upon the flight velocity change. The above results include, as a special case, $N = 1$, and there is no amplitude change.

Physically, amplitude effects arise from the relationship between the amplitude change due to the varying spring constant and the damping coefficient (quasi-steady terms) arising due to nonsteady flight.

It is interesting to note that if we approximate the solution to the differential equations V-67 and V-68 directly by asymptotic solutions we will obtain results quite similar to V-93 and V-94 for any velocity time history.

Returning to equations V-70 and V-71,

$$\frac{d^2 q}{d\tau_1^2} + \frac{1-N}{2} \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1} \frac{dq}{d\tau_1} + q = 0, \quad (V-70)$$

$$\frac{d^2 w}{d\tau_1^2} + \frac{N-1}{2} \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1} \frac{dw}{d\tau_1} + w = 0. \quad (V-71)$$

We transform the dependent variables by

$$q = Q e^{-\int \frac{1-N}{4} \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1} d\tau_1}, \quad (V-95)$$

$$w = W e^{-\int \frac{N-1}{4} \frac{1}{\bar{U}} \frac{d\bar{U}}{d\tau_1} d\tau_1}, \quad (V-96)$$

and then neglect the effect of the modifying terms in the restoring force term of the transformed differential equation to obtain:

$$q = (\bar{U})^{\frac{N-1}{4}} C_1 \cos(\tau_1 + \Phi_1), \quad (V-97)$$

$$w = (\bar{U})^{\frac{1-N}{4}} C_2 \cos (\tau_1 + \Phi_2), \quad (V-98)$$

or

$$q = (\bar{U})^{\frac{N-1}{4}} C_1 \cos \left[\int_0^{\tau} \bar{U}^{\frac{N+1}{2}} ds + \Phi_1 \right], \quad (V-99)$$

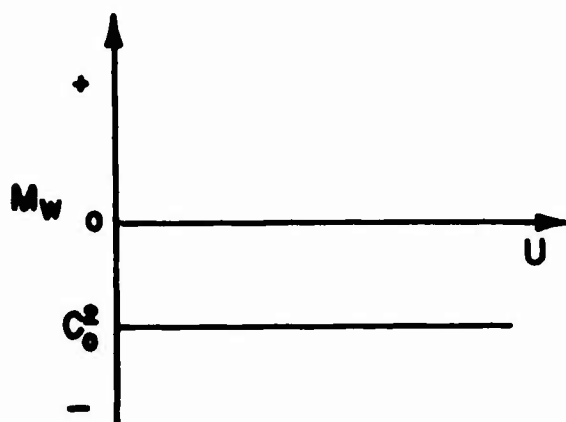
$$w = (\bar{U})^{\frac{1-N}{4}} C_2 \cos \left[\int_0^{\tau} \bar{U}^{\frac{N+1}{2}} ds + \Phi_2 \right], \quad (V-100)$$

applicable for any time history, as long as the initial value of \bar{U} is not too small. Comparison of equations V-99 and V-100 with V-93 and V-94 shows that the envelope is the asymptotic expansion of the Bessel functions, and the only real change is the appearance of the integral in the frequency term.

We present responses for two specific exponents so that the magnitude of the effects may be seen. We select the exponents zero and two, so as to present results on either side of the conventional case ($N = 1$).

The first example, where the exponent is zero, and therefore the attitude stability is independent of speed, might be considered a rough approximation to a tilt-wing VTOL aircraft, representing a limiting case. That is, if we imagine a tilt-wing aircraft with the horizontal tail completely immersed in the propeller slipstream, taken with the fact that through a transition the sum of the downwash velocity and the freestream velocity is approximately constant, the attitude stability would tend to be constant. The conventional airplane represents another limiting case where the horizontal tail is entirely out of the wake. These approximations do not include any effects from the wing-propeller combination or the fuselage. A form that includes these effects is considered later. We have no particular physical model in mind when the exponent is two.

1. Constant Attitude Stability



Attitude stability is independent of velocity, and there is no airframe damping. The differential equation for pitching velocity from equation V-65 with $N = 0$ is:

$$\frac{d^2 q}{dt^2} + C_o^2 U q = 0, \quad (V-101)$$

and the vertical velocity is determined simply from the moment equation V-18b,

$$w = - \frac{1}{C_o^2} \frac{dq}{dt}. \quad (V-102)$$

We again present specific results only for constant acceleration and deceleration. The characteristic time follows from equation V-66:

$$\tau = \omega_o (1 + f_o)^{\frac{1}{3}}$$

where

$$f = \frac{\dot{U}}{\omega U} = \frac{\dot{U}}{\sqrt{-M_w U^{\frac{3}{2}}}}.$$

In nondimensional form, the equation for pitching velocity is

$$\frac{d^2 q}{d\tau^2} + \bar{U} q = 0. \quad (V-103)$$

From equation V-72,

$$\bar{U} = \left[\frac{1}{(1 + f_o)^{\frac{2}{3}}} + \frac{f_o}{1 + f_o} \tau \right].$$

The limiting case of $f_o = 0$ results in a constant coefficient equation with unity frequency. The value of f_o determines the value of the argument when $\tau = 0$.

Figure 15 shows the pitching velocity and vertical velocity response for various values of the acceleration parameter for initial conditions $q(0) = 1$, $w(0) = 0$.

As would be expected, the frequency change is smaller than the conventional case (cf. Figure 3).

When f_o is positive, the amplitude of the pitching velocity decreases with time, and the amplitude of the vertical velocity increases with time. When f_o is negative, the variations are reversed.

The apparent damping ratio is not large, so the changes in the amplitude of the response due to variable velocity would be most evident when the airplane has little or no inherent damping.

2. Attitude Stability Proportional to the Square of the Velocity

The characteristic velocity and time, from V-66, are:

$$U^* = U_o (1 + f_o)^{\frac{2}{5}},$$

$$\tau = \omega_o (1 + f_o)^{\frac{3}{5}} t. \quad (V-104)$$

Figure 16 shows responses for various values of f_o . Here, as expected, there is a rapid increase in frequency with time, as well as a reverse in the amplitude variation with time of the two variables compared to the case $N = 0$. When f_o is positive, the amplitude of the pitching velocity increases with time, and the amplitude of the vertical velocity decreases with time. Physically, the effect of the quasi-steady

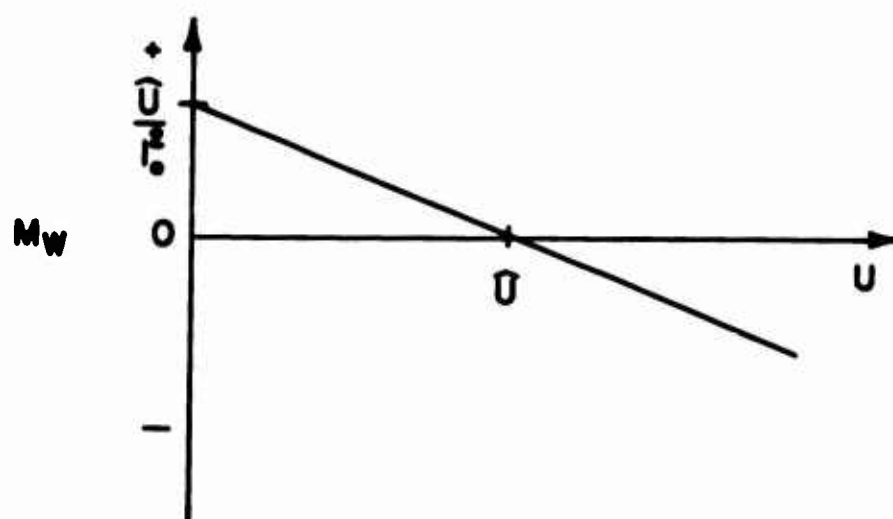
damping term in the pitching velocity equation V-65, $\frac{\dot{M}_w}{M_w}$, has increased more rapidly than the effect of spring constant variation on the amplitude of the motion. In the vertical velocity equation, V-64, there is no change in the quasi-steady damping term equal to $\frac{\dot{U}}{U}$, while the rate of increase of the spring constant has become larger, causing the apparent damping due to spring constant variation to be more powerful than the direct effect of $\frac{U}{U}$.

Thus we have a good indication of the importance of unsteady effects and how they depend upon the manner in which the attitude stability varies with velocity.

We investigate at low speeds one other specific case that approximates the variation of the attitude stability of VTOL aircraft with speed.

b. Attitude Stability Varies Linearly With Velocity, and is Nonzero at Hover

Typically, for a VTOL aircraft at low speeds, the attitude stability is positive (unstable) in hovering, and as the flight speed increases, the attitude stability changes sign, as shown in the following sketch (Appendix III).



\hat{U} denotes the velocity at which attitude stability changes sign.

In steady flight at velocities less than \hat{U} , the modes of motion, with no damping, are a convergence and a divergence, and at velocities greater than \hat{U} are neutrally stable oscillation.

We use nondimensional forms such that this case will reduce to the conventional aircraft case as \hat{U} approaches zero.

We now have difficulty obtaining analytical solutions or using asymptotic expressions, due to the fact that the restoring moment coefficient has a zero in the time interval of interest. Therefore, these solutions were obtained using a digital computer.

The differential equations are:

$$\frac{d^2 q}{dt^2} + \frac{\dot{U}}{\hat{U} - U} \frac{dq}{dt} + \frac{1}{l_o^2} (U - \hat{U}) U q = 0, \quad (V-105)$$

$$\frac{d^2 w}{dt^2} - \frac{\dot{U}}{U} \frac{dw}{dt} + \frac{1}{l_o^2} (U - \hat{U}) U w = 0, \quad (V-106)$$

where

$$M_w = \frac{1}{l_o^2} (\hat{U} - U). \quad (V-107)$$

We nondimensionalize these equations with the following characteristic velocity and time:

$$\begin{aligned} U^* &= U_o (1 + f_o)^{\frac{1}{2}} \\ \tau &= \frac{U^*}{l_o} t \\ f_o &= \frac{\dot{U}_o l_o}{U_o^2}, \end{aligned} \quad (V-108)$$

where as $U_o \rightarrow 0$,

$$\begin{aligned} U^* &\rightarrow \sqrt{\dot{U}_o l_o} \\ \tau &\rightarrow \sqrt{\frac{\dot{U}_o}{l_o}}. \end{aligned}$$

We define a new parameter as:

$$\hat{f} = \frac{\dot{U}_0 l_0}{\hat{U}^2} \quad (V-109)$$

These parameters are well behaved for all values of U_0 , and reduce to the conventional case as $U \rightarrow 0$ ($f \rightarrow \infty$).

$$\frac{d^2 q}{d\tau^2} + \frac{\frac{d\bar{U}}{d\tau}}{\bar{U} - \hat{U}} \frac{dq}{d\tau} + (\bar{U} - \hat{U}) \bar{U} q = 0, \quad (V-110)$$

$$\frac{d^2 w}{d\tau^2} - \frac{\frac{d\bar{U}}{d\tau}}{\bar{U}} \frac{dw}{d\tau} + (\bar{U} - \hat{U}) \bar{U} w = 0, \quad (V-111)$$

where \hat{U} is a constant,

$$\hat{U} = \left(\frac{f_0}{\hat{f}} \right)^{\frac{1}{2}} \frac{1}{\sqrt{1+f_0}}. \quad (V-112)$$

For constant acceleration,

$$\bar{U} = \left[\frac{1}{\sqrt{1+f_0}} + \frac{f_0}{1+f_0} \tau \right]; \quad (V-113)$$

as $U_0 \rightarrow 0$, $f_0 \rightarrow \infty$,

$$\hat{U} \rightarrow \frac{1}{\hat{f}^{\frac{1}{2}}} \quad \bar{U} \rightarrow \tau.$$

Equations V-110 and V-111 may be written in an alternate form:

$$\frac{d^2 q}{d\tau^2} + \frac{\frac{d\bar{U}}{d\tau}}{\frac{f_0}{\sqrt{\hat{f}(1+f_0)}} - \bar{U}} \frac{dq}{d\tau} + \left[\bar{U} - \sqrt{\frac{f_0}{\hat{f}(1+f_0)}} \right] \bar{U} q = 0, \quad (V-114)$$

$$\frac{d^2 w}{d\tau^2} - \frac{\frac{d\bar{U}}{d\tau}}{\bar{U}} \frac{dw}{d\tau} + \left[\bar{U} - \sqrt{\frac{f_o}{\hat{f}(1+f_o)}} \right] \bar{U} w = 0. \quad (V-115)$$

The parameter \hat{f} may be interpreted in the following way:

$$\hat{f} = \frac{\dot{U}_o l_o}{\hat{U}^2} \sqrt{\frac{f_o}{f}} = \frac{\hat{U}}{U_o}.$$

For constant acceleration, $\frac{\hat{U}}{U_o}$ is the time required for the airplane to accelerate from hovering to the velocity \hat{U} . $\frac{\hat{U}}{l_o}$ is the frequency of the oscillatory motion at a steady velocity, $U = \left(\frac{1+\sqrt{5}}{2}\right) \hat{U}$.

Thus,

$$\hat{f} = \left(\frac{1}{T_{U=\hat{U}}} \frac{P}{2\pi} U \right) = \left(\frac{1+\sqrt{5}}{2} \right) \hat{U}.$$

As \hat{f} becomes large, $M_w \rightarrow 0$ when $U = 0$. This is the situation when the time required for the aircraft to reach the velocity \hat{U} is short compared to the period of the motion at $U = \left(\frac{1+\sqrt{5}}{2}\right) \hat{U}$. As the time interval during which M_w is positive becomes shorter, the effect of the instability on the response is less significant. As the time interval becomes long, the effects of the unstable region become more pronounced. These effects are clearly shown in Figure 17 where the pitching velocity response is shown vs. τ for various values of \hat{f} . $\hat{f} = \infty$ is included for comparison to the conventional case.

From equations V-114 and V-115 it may be seen that as long as $\frac{f_o}{\hat{f}(1+f_o)}$ is constant, the differential equations will be the same in non-dimensional time. For example, the curve given for $\hat{f} = .5$, $f_o = \infty$, will also be the response when $\hat{f} = .25$, $f_o = 1$, and $\hat{f} = .0833$, $f_o = .2$. Thus there is an equivalence in increasing the initial flight velocity from zero, and at the same time increasing the velocity \hat{U} . However, the time scale will change, due to its dependence on f_o .

The nondimensional time at which the attitude stability (M_w) equals zero is given by

$$\tau_{CR} = \frac{\sqrt{1+f_o}}{f_o} \left[\sqrt{\frac{f_o}{f}} - 1 \right], \quad (V-116)$$

as $f_o \rightarrow \infty$

$$\tau_{CR} \rightarrow \frac{1}{\sqrt{f}}$$

This time is indicated on Figure 17.

c. Effects of Airframe Damping

We now examine the effects of airframe damping on the preceding examples. A transformation that is useful is discussed, and then some general features and specific examples are given.

The differential equations for pitching velocity and vertical velocity are from Appendix II:

$$\ddot{w} + \left[-Z_w - M_q - \frac{\dot{U}}{U} \right] \dot{w} + \left[-UM_w + Z_w M_q + \dot{Z}_w + \frac{\dot{U}}{U} Z_w \right] w = 0, \quad (V-117)$$

$$\ddot{q} + \left[-Z_w - M_q - \frac{\dot{M}_w}{M_w} \right] \dot{q} + \left[-UM_w + Z_w M_q + M_q \frac{\dot{M}_w}{M_w} + \dot{M}_q \right] q = 0. \quad (V-118)$$

In order to determine how airframe damping influences the solution, the following transformation will elucidate the possibilities that may arise.

We will select for detailed comment only the most critical case. We restrict discussion to aircraft with stable damping ($Z_w < 0$, $M_q < 0$).

The short period equations are:

$$\frac{dw}{dt} = Z_w w + U q, \quad (18a)$$

$$\frac{dq}{dt} = M_q q + M_w w. \quad (18b)$$

$$\text{Let } q = Q(t) e^{\frac{1}{2} \int_0^t (Z_w + M_q) ds} \quad w = W(t) e^{\frac{1}{2} \int_0^t (Z_w + M_q) ds}, \quad (\text{V-119})$$

and define $A = \frac{Z_w - M_q}{2}$, and the transformed short period equations become:

$$\frac{dW}{dt} = A W + U Q, \quad (\text{V-120a})$$

$$\frac{dQ}{dt} = -A Q + \dot{M}_w W. \quad (\text{V-120b})$$

Therefore, the differential equations for Q and W are:

$$\frac{d^2 Q}{dt^2} - \frac{\dot{M}_w}{M_w} \frac{dQ}{dt} - [M_w U + A^2 + \frac{\dot{M}_w}{M_w} A - \dot{A}] Q = 0, \quad (\text{V-121})$$

$$\frac{d^2 W}{dt^2} - \frac{\dot{U}}{U} \frac{dW}{dt} - [M_w U + A^2 - \frac{\dot{U}}{U} A + \dot{A}] W = 0. \quad (\text{V-122})$$

We can distinguish three possibilities for approximations to equations V-121 and V-122.

1. When $\left| A^2 + \frac{\dot{M}_w}{M_w} A - \dot{A} \right| \gg \left| M_w U \right|$, V-121 and V-122 have exact solutions:

$$Q = e^{-\int_0^t A ds} \left[C_1 + C_2 \int_0^t M_w(s) e^{2 \int_0^s A dn} ds \right], \quad (\text{V-123})$$

$$W = e^{\int_0^t A ds} \left[C_3 + C_4 \int_0^t U(s) e^{-2 \int_0^s A dn} ds \right], \quad (\text{V-124})$$

or in terms of the stability derivatives and the aircraft variables,

$$q(t) = e^{\int_0^t M_q ds} \left[C_1 + C_2 \int_0^t M_w(s) e^{\int_0^s (Z_w - M_q) dn} ds \right], \quad (\text{V-125})$$

$$w(t) = e^{\int_0^t Z_w ds} \left[C_3 + C_4 \int_0^t U(s) e^{\int_0^s (M_q - Z_w) dn} ds \right]. \quad (V-126)$$

In this case, the two degrees of freedom, q and w , are weakly coupled. This solution is equivalent to considering the variables dynamically uncoupled. If M_w is zero, then the integral in the first equation would not be present, resulting in a first-order uncoupled response describing the pitching velocity. Then the second term in equation V-126 arises from the fact that any motion in q will be reflected by the term $U q$ in the vertical force equation V-18a. Then, we may interpret the integral term in equation V-125 in a similar fashion. Thus, the form of the solution represents solving the uncoupled first-order equations describing pitching velocity and vertical velocity, and then considering these responses as disturbances upon each other, due to the coupling terms.

If M_q and Z_w are appreciable then, these responses would represent over critically damped cases. This result would be typical of a vehicle with heavy damping and is not considered to be a critical case. If the damping is light, then the period is also long (M_w very small) and the dynamic motion is slow, or the aerodynamic forces on the airplane may be neglected.

$$2. \quad \left| M_w U \right| \gg \left| A^2 + \frac{\dot{M}_w}{M_w} A - \dot{A} \right|.$$

The transformed differential equations are

$$\frac{d^2 Q}{dt^2} - \frac{\dot{M}_w}{M_w} \frac{dQ}{dt} - (M_w U) Q = 0, \quad (V-127)$$

$$\frac{d^2 W}{dt^2} - \frac{\dot{U}}{U} \frac{dW}{dt} - (M_w U) W = 0. \quad (V-128)$$

The effects of airframe damping appear only as an amplitude modification through the original transformation (compare V-127 and V-128 with V-64 and V-65 when $N = 1$). Thus, to determine the influence of the damping we need only to add the aerodynamic damping effects to the unsteady effects. The frequency of the motion ($M_w U$) is of a significant magnitude, and the airplane damping is not large, so this would be a critical situation.

Note that this case is exact when $Z_w = M_q$, which in many instances is approximately true.

3. The third case is the situation in which the terms involving A are of similar magnitude to those of $M_w U$. This would represent the transition from the uncoupled situation discussed under case 1 and case 2. This is a reasonably well damped case and thus is not a particularly critical situation and will not be considered further. This situation would most likely be significant when either M_q or Z_w are zero and the transformation V-119 is not useful, and either V-117 or V-118 will be simplified.

Let us proceed to examine case 2 in some detail where we can add the damping effects directly to the previous results; i. e., the solutions to V-127 and V-128 with the transformations V-119 are:

$$q = e^{\frac{1}{2} \int_0^t (Z_w + M_q) ds} \quad [\text{solution with no damping}], \quad (\text{V-129})$$

$$w = e^{\frac{1}{2} \int_0^t (Z_w + M_q) ds} \quad [\text{solution with no damping}]. \quad (\text{V-130})$$

Now nondimensionalize the exponential term due to damping. We recall that

$$\tau = C_o (U^*)^{\frac{N+1}{2}} t, \quad (\text{V-66})$$

so that the exponential term may be written

$$e^{\frac{1}{2} \int_0^t \frac{(Z_w + M_q)}{2 C_o U^* \frac{N+1}{2}} ds} \quad (\text{V-131})$$

Now we define

$$\delta_B = \frac{-(Z_w + M_q)}{2 \sqrt{-M_w U}},$$

and recall that

$$-M_w = C_o^2 U^N \quad (V-63)$$

and that

$$\frac{d\tau_1}{d\tau} = (\bar{U})^{\frac{N+1}{2}}, \quad (V-73)$$

so that in terms of the stretched time, τ_1 , equation V-131 becomes

$$e^{-\int_0^{\tau_1} \delta_B(s) ds} \quad (V-132)$$

for the exponential term, so that the solutions V-129 and V-130 may be written as:

$$q = e^{-\int_0^{\tau_1} \delta_B(s) ds} \quad [\text{solution with no damping}], \quad (V-133)$$

$$w = e^{-\int_0^{\tau_1} \delta_B(s) ds} \quad [\text{solution with no damping}]. \quad (V-134)$$

We are now in a position to evaluate the complete solution.

In order to estimate the importance of the aerodynamic damping compared to the unsteady effects, it is particularly enlightening to consider the asymptotic solutions expressed in terms of $f(\tau_1)$.

We have previously obtained equations in the transformed time scale, in terms of the acceleration parameter $f(\tau_1)$:

$$\frac{d^2 Q}{d\tau_1^2} + \left(\frac{1-N}{2}\right) f(\tau_1) \frac{dQ}{d\tau_1} + Q = 0, \quad (V-70a)$$

$$\frac{d^2 W}{d\tau_1^2} + \left(\frac{N-1}{2}\right) f(\tau_1) \frac{dW}{d\tau_1} + W = 0. \quad (V-71a)$$

The asymptotic solutions to these equations are:

$$q = e^{-\int_0^{\tau_1} \left[\frac{1-N}{4} f(s) \right] ds} C_1 \cos(\tau_1 + \Phi_1), \quad (V-135)$$

$$W = e^{-\int_0^{\tau_1} \left[\frac{N-1}{4} f(s) \right] ds} C_2 \cos(\tau_1 + \Phi_2), \quad (V-136)$$

and so the complete solution from V-133 and V-134 is:

$$q = e^{-\int_0^{\tau_1} \left[\delta_B(s) + \frac{1-N}{4} f(s) \right] ds} C_1 \cos(\tau_1 + \Phi_1), \quad (V-137)$$

$$w = e^{-\int_0^{\tau_1} \left[\delta_B(s) + \frac{N-1}{4} f(s) \right] ds} C_2 \cos(\tau_1 + \Phi_2). \quad (V-138)$$

While the term $(f(s) ds)$ is an exact differential (cf. V-137, V-138 with V-99 and V-100 when $\delta_B \equiv 0$), it is perhaps more convenient to think of the results in the above terms. Largely, then, the importance of the unsteady flight condition in causing changes in the apparent damping can be thought of in two ways. First, the above results indicate directly a way in which to estimate the importance of damping alterations due to unsteady flight by comparing the magnitude of $f(\tau)$ throughout the interval of interest with the magnitude of the instantaneous value of the damping ratio $\delta_B(\tau)$. Both quantities are nondimensional, so that they may be computed in real time and compared on that basis. Their variation with time will be different, depending upon whether they are computed as a function of real time, or the stretched time, τ .

The second effect, that is the only effect present when $N = 1$ (the conventional aircraft), is due to the distortion of the time scale discussed in some detail in Section 1 of this chapter.

When asymptotic solutions are not valid, then it is not so easy to generalize; however, the previous results would lead us to believe that the effects would not be large or unusual.

One other aspect of the usefulness of the asymptotic solutions is worth mentioning. Consider the untransformed equations V-117 and V-118:

$$\frac{d^2 q}{dt^2} + \left[-Z_w - M_q - \frac{\dot{M}_w}{M_w} \right] \frac{dq}{dt} + \left[-M_w U + D_q \right] q = 0, \quad (V-113)$$

$$\frac{d^2 w}{dt^2} + \left[-Z_w - M_q - \frac{\dot{U}}{U} \right] \frac{dw}{dt} + \left[-M_w U + D_w \right] w = 0, \quad (V-114)$$

where

$$D_w = Z_w M_q + \dot{Z}_w + \frac{\dot{U}}{U} Z_w, \quad (V-139)$$

$$D_q = Z_w M_q + M_q \frac{\dot{M}_w}{M_w} + \dot{M}_q. \quad (V-140)$$

The nondimensionalized equations, when $M_w = -C_o^2 U^N$, are:

$$\frac{d^2 q}{d\tau_1^2} + \left[2\delta_B(\tau_1) + \left(\frac{1-N}{2} \right) f(\tau_1) \right] \frac{dq}{d\tau_1} + \left[1 + \frac{D_q}{M_w U} \right] q = 0, \quad (V-141)$$

$$\frac{d^2 w}{d\tau_1^2} + \left[2\delta_B(\tau_1) + \left(\frac{N-1}{2} \right) f(\tau_1) \right] \frac{dw}{d\tau_1} + \left[1 + \frac{D_w}{M_w U} \right] w = 0, \quad (V-142)$$

so in cases where $f(\tau_1)$ is large, it still may be possible to use the asymptotic solutions to V-141 or V-142 to describe the response if either

$$2\delta_B(\tau_1) + \left(\frac{1-N}{2} \right) f(\tau_1)$$

or

$$2\delta_B(\tau_1) - \left(\frac{1-N}{2} \right) f(\tau_1)$$

are small, if the terms $\frac{D_w}{M_w U}$ and $\frac{D_q}{M_w U}$ are small compared to 1.

Thus, it is possible to estimate the importance of unsteady effects, at least in the transformed plane, by comparing the magnitudes of $\delta_B(\tau_1)$ or $\delta_B(t)$ to $f(\tau_1)$, $[f(t)]$, even when $f(\tau_1)$ is large, if the sum of the aerodynamic damping and this term are small.

The distortion of the time scale due to the independent variable transformation is most easily visualized by the method suggested earlier of drawing a curve of the relationship between τ_1 and τ , and noting the deviation from the line with a slope of 1, representing no distortion arising from the transformation.

d. Other Stability Derivative Variations

We have discussed specific stability derivative variations in order to obtain an estimate of the magnitude and importance of the effects of variable flight velocity on the short period motion. We can proceed in a similar fashion to analyze any stability derivative variations using the approximate approach on the complex plane developed in Chapter IV.

Qualitatively the important effects arise in two ways. We restrict our discussion to motion that is oscillatory. The primary changes from the frozen system arise from the varying spring constant and give rise to:

1. A distortion of the time scale of the motion.
2. A contribution to the damping of the motion.

They may be viewed as resulting from the two transformations leading to the asymptotic approximation. The actual form of the acceleration parameter will be slightly different when more general variations in attitude stability are considered. The equation for pitching velocity is V-118:

$$\frac{d^2 q}{dt^2} + [2\sigma'] \frac{dq}{dt} + [\omega'^2] q = 0, \quad (V-143)$$

where the primes indicate that these coefficients are different at any instant of time than the frozen system coefficients. We then transform the time by

$$\tau_1 = \int_0^{\tau} \omega' ds \quad (V-144)$$

to obtain

$$\frac{d^2 q}{d\tau_1^2} + \left[\frac{2\sigma'}{\omega'} + \frac{1}{2} \frac{\omega'}{\omega' \frac{3}{2}} \right] \frac{dq}{d\tau_1} + q = 0. \quad (V-145)$$

The first effect arises from the transformation, V-144, and the second effect of importance arises from the appearance of the fractional rate of change of the frequency appearing in the damping term in equation V-145. Since the first term may be interpreted as the instantaneous damping ratio, then the size and importance of the second effect may be estimated by comparing the fractional rate of change of the frequency with the instantaneous damping ratio.

When M_w dominates the restoring force term, as is generally the case when the motion is less than critically damped, the unsteady parameter is:

$$\frac{\dot{\omega}}{\omega \frac{3}{2}} = f \left[1 + \frac{U}{M_w} \frac{d M_w}{d U} \right].$$

When the approximate solutions do not apply, it becomes more difficult to draw general conclusions about the nature of the response.

The approximate representation will describe with reasonable accuracy the behavior of the vehicle wherever the transient motion is on the borderline between stability and instability, the case of critical interest.

For the case where the problem of interest is such that the total coefficient change is constant, and occurs in different time intervals, we may place limits on the dynamic behavior for very rapid coefficient changes by using the approximate solution obtained by neglecting the restoring force term.

It must be carefully noted that the above comments apply only to the nature of the time histories of pitching velocity and vertical velocity, and any other variables such as angle of attack and normal acceleration must be considered as well. The effects on these variables due to unsteady flight are more pronounced than the effects upon the pitching velocity and vertical velocity and must be investigated to provide a complete understanding of the motion of the airplane in unsteady flight.

C. Higher-Order Systems

We now investigate the nature and magnitude of the unsteady effects on a specific higher-order system. In Section B a detailed investigation of the short period motion has been made.

We will use the approximate method developed in Chapter IV to compute the magnitude of the effects. The third-order system describing the attitude-velocity motion will be investigated and should provide a good description of the longer time response of VTOL aircraft at low speeds.

When studying the longer time response, one might expect larger changes in the character of the motion from varying flight velocity, due to the fact that the parameter indicating the importance of unsteady effects, $f = \frac{U}{\omega U}$, becomes larger as the "frequency" of the motion, ω , becomes smaller. In the motions of a conventional aircraft, for example, $|f|$ based on the short period frequency will be considerably less than $|f|$ based on the phugoid frequency. While this is true, alterations in the character of the phugoid will not be of so much significance to piloting the aircraft as will alterations in the short period motion, since the motions are slower. Phugoid characteristics are of more importance in the cruising flight of an aircraft, when instabilities are primarily fatiguing to the pilot and not dangerous.

For VTOL aircraft in particular, we are not in a position to obtain the results in quite so general a form as in the case of the conventional airplane, due to the relatively complicated nature of the stability derivatives of these vehicles. We will use, as numerical examples, stability derivatives that are relatively typical of VTOL aircraft as determined from experiments at Princeton and elsewhere. The stability derivatives and their expected behavior are discussed in some detail in Appendix III, on the basis of experimental results presently available.

We now proceed to the investigation of the two-degree-of-freedom motion typical of VTOL aircraft near hovering. This description is exact when $M_w = 0$ and $X_w = 0$.

The differential equations describing the linearized dynamics of a VTOL aircraft at slow speeds and in hovering (Reference 14) are:

$$X_u u - \dot{u} - g \theta = 0, \quad (V-146a)$$

$$M_u u + M_q \dot{\theta} - \ddot{\theta} = 0, \quad (V-146b)$$

where for simplicity the usually small term X_q has been neglected. As discussed in Appendix III, the stability derivatives are complicated functions of power setting, wing tilt angle, and flight speed. It seems a reasonable approximation to consider the pitch damping as only a function of velocity, since the horizontal tail produces the primary contribution to this derivative. It is perhaps more plausible to consider the velocity derivatives as only functions of wing tilt angle. This is equivalent to assuming that the velocity derivatives are linear, a reasonable assumption except for rapid decelerations, where the wing, a primary contributor to these derivatives, may stall, for reasons discussed in Appendix I. For a first approach, we will not take into account the differences in these functional dependencies, but will consider all three derivatives as a function of a single parameter, velocity in this case. This approximation and its implications are discussed in more detail in Appendix III.

We now proceed to investigate the nature of the unsteady effects on this motion, using the approximate methods of Chapter IV.

First, we determine the differential equations describing the time history of the two variables θ and u from V-146a and V-146b.

We recall that the differential equations for each variable will be different, due to the stability derivatives changing with time (velocity).

Then the differential equations for u and θ are:

$$\frac{d^3\theta}{dt^3} + \left[-X_u - M_q - \frac{1}{M_u} \frac{d(M_u)}{dt} \right] \frac{d^2\theta}{dt^2} + \left[M_q X_u - \frac{d(M_q)}{dt} + \frac{M_q}{M_u} \frac{d(M_u)}{dt} \right] \frac{d\theta}{dt} + g M_u \theta = 0, \quad (V-147)$$

$$\frac{d^3u}{dt^3} + \left[-X_u - M_q \right] \frac{d^2u}{dt^2} + \left[M_q X_u - 2 \frac{dX_u}{dt} \right] \frac{du}{dt} + \left[g M_u - \frac{d^2(X_u)}{dt^2} + M_q \frac{dX_u}{dt} \right] u = 0, \quad (V-148)$$

where the underlined terms are new terms due to unsteady flight. When the flight condition is steady, these terms are zero, and the differential equations V-147 and V-148 are identical.

These additional terms give rise to what we have referred to as quasi-steady effects. That is, changes in the coefficients of the differential equations compared to the steady case. Again, this is a system with strong coupling; i. e., the frequency of the oscillatory motion is primarily determined by $(g M_u)$, and so the quasi-steady effects may be a poor reflection of the total unsteady effects. It is the quasi-steady effects that give rise to the differences in the time histories of different variables.

The source of differences between these two variables, indicated by the presence of additional terms depending upon the rate of change of the stability derivatives with time, may be interpreted if we consider the case in which X_u is zero. Then the relationship between u and θ from V-146a is:

$$u = -g \int_0^t \theta \, ds.$$

When $X_u = 0$, the differential equation V-147 may be integrated with respect to time, resulting in:

$$\frac{d^3}{dt^3} \left(\int_0^t \theta \, ds \right) - M_q \frac{d^2}{dt^2} \left(\int_0^t \theta \, ds \right) + g M_u \int_0^t \theta \, ds = 0, \quad (V-149)$$

and the equation for u , when $X_u = 0$, is from V-148,

$$\frac{d^3 u}{dt^3} - M_q \frac{d^2 u}{dt^2} + g M_u u = 0. \quad (V-150)$$

The differential equations for $\int_0^t \theta \, ds$ and u are identical, even in the unsteady case.

Thus $\int_0^t \theta \, ds$ and $u(t)$ are composed of the same time functions when X_u is equal to zero. Since these are time variable equations, however, differentiation to obtain θ will result in different functions of time. For the oscillatory mode, for example, if the frequency is increasing, the amplitude of θ will grow at a faster rate than $\int_0^t \theta \, ds$, and consequently

also at a faster rate than u . The converse will be true if the frequency is decreasing. This difference is reflected by the additional terms in the coefficients of the differential equation for θ (compare V-149 to V-147, with $X_u = 0$).

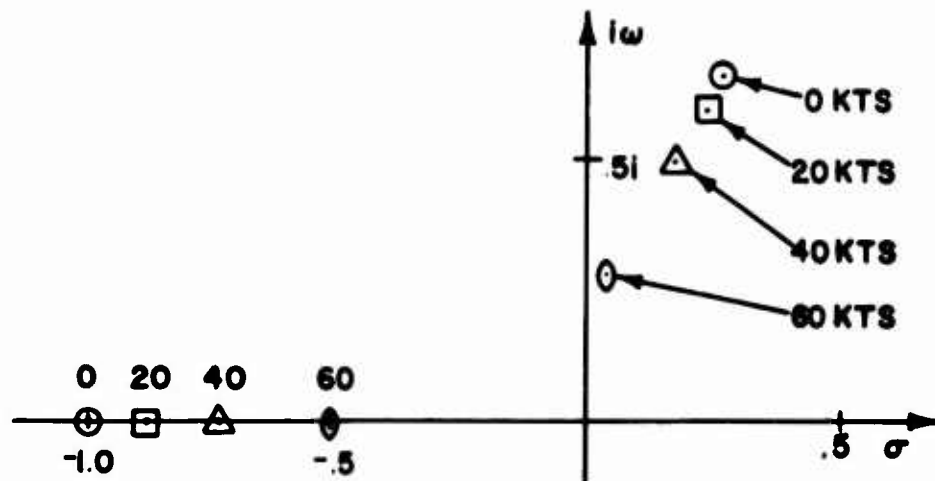
First we consider the nature of the solutions of equation V-147 for pitch angle. The new terms are $\frac{d(M_u)}{dt}$ and $\frac{d(M_q)}{dt}$, the first term appearing as a ratio, i. e. $\frac{1}{M_u} \frac{d(M_u)}{dt}$, due to the fact that it is a coupling term (cf. equation III-13). As this quantity becomes very large, indicating small values of M_u , the differential equation for θ approaches $\frac{d^2\theta}{dt^2} + M_q \frac{d\theta}{dt} = 0$, the differential equation describing the pitching motion when $M_u = 0$.

The quasi-steady roots show the influence of these additional terms on the dynamics of the system. The root locus equation to determine the quasi-steady effects on pitch angle is obtained by considering equation V-147 as a constant coefficient equation and therefore assuming that $\frac{d\theta}{dt} = r\theta$, $\frac{d^2\theta}{dt^2} = r^2\theta$ and $\frac{d^3\theta}{dt^3} = r^3\theta$. Rearranging,

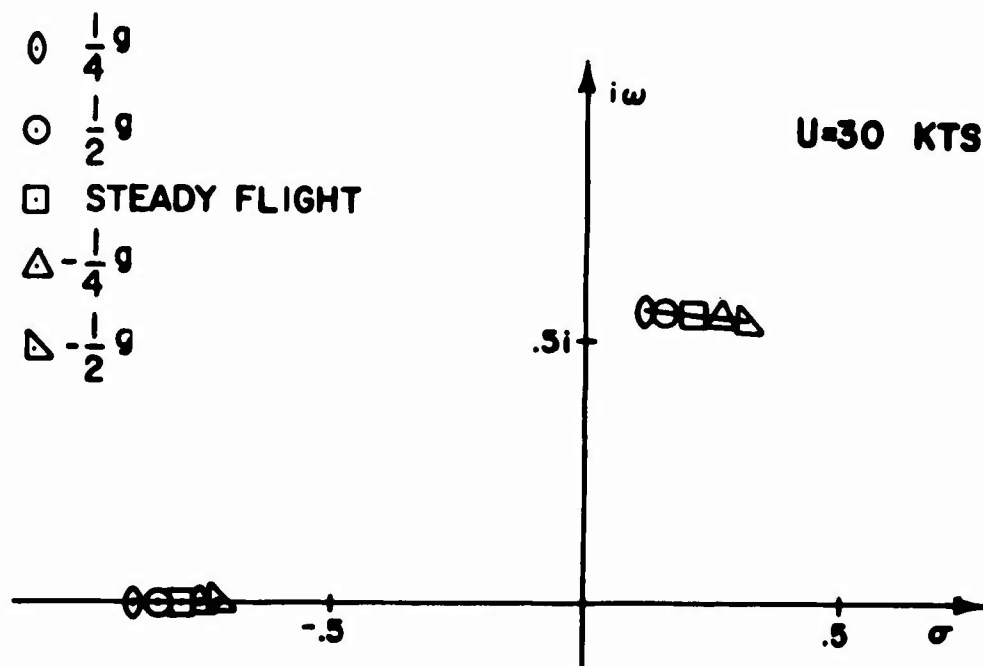
$$\frac{-\frac{1}{M_u} \frac{d M_u}{dt} \left\{ r \left[r - M_q - M_q \frac{M_u}{M_u} \right] \right\}}{r^3 + [-X_u - M_q] r^2 + [M_q X_u] r + g M_u} = -1. \quad (V-151)$$

A 180° locus is required when $\frac{1}{M_u} \frac{d(M_u)}{dt}$ is negative, and a zero degree locus when $\frac{1}{M_u} \frac{d(M_u)}{dt}$ is positive. Typically, M_u is positive at low speeds, and therefore the 180° locus will represent the case where M_u is decreasing with time, and the 0° locus M_u is increasing with time for an accelerating transition.

A typical frozen locus of roots near hovering (stability derivatives typical of a tilt-wing airplane of 40,000 pounds gross weight are used; the derivatives are given in Figure 28) is:



Then the effects of the quasi-steady terms will be as shown below, using the frozen roots as poles for equation V-151:



A decreasing M_u with time, typical of increasing speed from hovering, causes an increase in apparent damping of the oscillatory mode, and an increasing M_u , as experienced in deceleration, causes an apparent loss in damping of the pitch angle response.

This is the first step in obtaining the approximate solution. There is a further modification to this locus due to the unsteady effects; i. e., the actual change in the nature of the solutions due to varying coefficients. We use the modified root locus technique discussed in Chapter IV to display these effects; denoting the quasi-steady characteristic equation as:

$$\Delta = \lambda^3 + C_0' \lambda^2 + C_1' \lambda + C_2' \quad (V-152)$$

The unsteady effects are shown by the root locus drawn using equation:

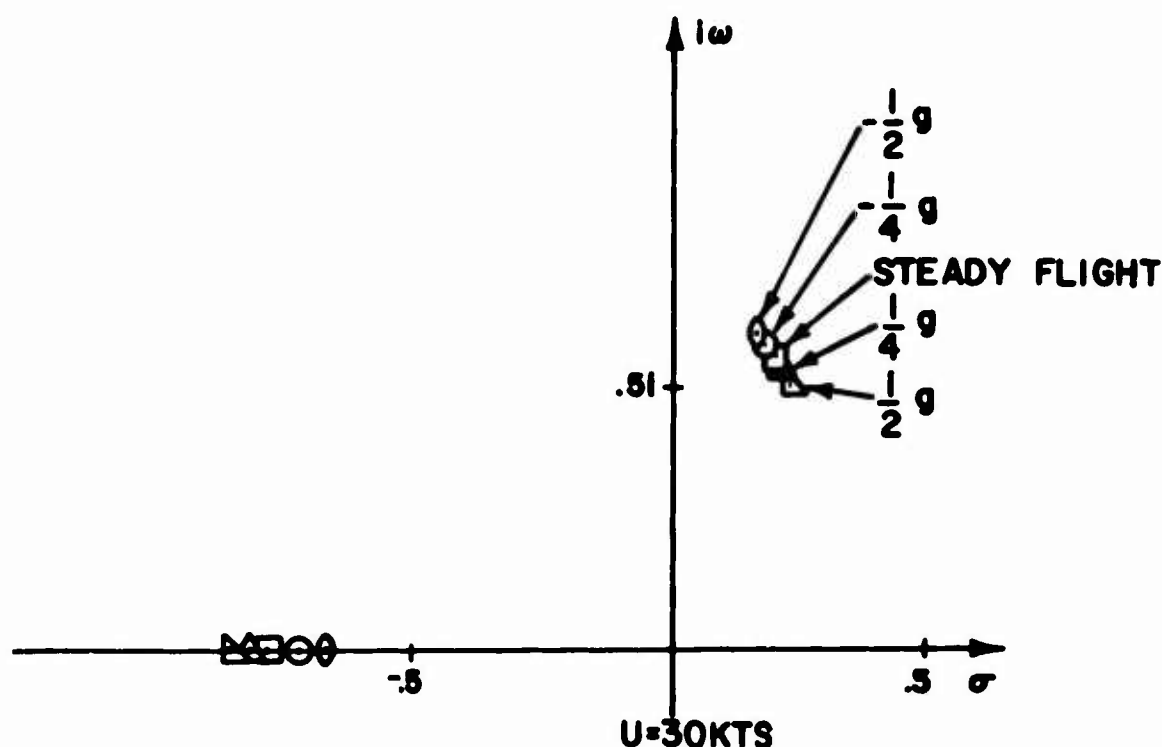
$$\frac{3\dot{r}(\lambda + \frac{C_o'}{3})}{\lambda^3 + C_o' \lambda^2 + C_1' \lambda + C_2'} = -1, \quad (V-153)$$

where the gain of the root locus is \dot{r} , in general, a complex number, so that we have other than the usual 180° or zero degree angle conditions. Since the quasi-steady roots change we do not have a continuous locus for the unsteady roots. That is, the poles of equation V-153 are dependent upon the root velocity. Only one point on each unsteady locus will be valid. Figure 19 shows the locus of unsteady roots for this case. Comparison of the unsteady roots shown in Figure 19 and the quasi-steady roots shown in Figure 18 indicates that the unsteady effects are opposite to the quasi-steady effects and roughly cancel for the pitch angle response. The general tendency of the unsteady effects on the oscillatory mode is an apparent loss in damping with decreasing frequency, and an apparent gain in damping with increasing frequency, as noted for the second-order system.

We then investigate the time-varying effects on the velocity perturbation in a similar fashion. The quasi-steady effects are displayed first, by constructing a root locus based on the equation V-150, obtained from V-148, assuming that $\frac{du}{dt} = r u$, $\frac{d^2 u}{dt^2} = r^2 u$, and $\frac{d^3 u}{dt^3} = r^3 u$:

$$\frac{-2 \frac{d(X_u)}{dt} \left[r - \frac{M_q}{2} + \frac{1}{2 \frac{dX_u}{dt}} \frac{d^2 X_u}{dt^2} \right]}{r^3 + C_o r^2 + C_1 r + C_2} = -1, \quad (V-154)$$

and obtain



as a modification to the frozen roots. In the usual case, X_u decreases with velocity and \ddot{X}_u is negative; the accelerating case would be reflected by a loss in damping of the oscillatory mode, and the decelerating case by an increase in damping, as well as a change in frequency as shown.

Now then, we evaluate the unsteady effects by using the quasi-steady roots as poles, and the unsteady "characteristic equation" is

$$\frac{3\ddot{x}[\lambda + \frac{C_0''}{3}]}{\lambda^3 + C_0''' \lambda^2 + C_1'' \lambda + C_2''} = -1, \quad (V-155)$$

where we have the same form for the locus as for θ , but the starting roots are now the quasi-steady roots for u . Here we see that in the typical case shown, the unsteady effects will cause the unsteady locus to depart even more from the frozen locus in contrast to the θ motion where the two effects were in opposite directions (cf. Figures 18 and 19). Figures 18 and 19 show the two complete loci separately; first, the quasi-steady loci, and then the unsteady loci, as discussed.

We note that the unsteady locus for pitch angle essentially indicates only a small change in the unsteady "roots" from the frozen roots; except

at the low frequency part of the locus where the departures are quite large, stretching the validity of the first-order theory. The locus for the velocity perturbation "roots" shows a definite improvement in the stability of the response in a decelerating transition and a loss in stability in the accelerating case.

Figure 20 shows response time histories computed by numerical integration using a digital computer compared to the envelope of the oscillatory motion based on initial values of the frozen roots. It can be seen that the trends indicated by the approximate root loci show up quite clearly.

In the case of the pitch angle response in the accelerating case, the instability in the response is not quite as large as the frozen case, due to the decreasing exponential term.

The envelope of the velocity perturbation remains quite close to the frozen envelope, as indicated by a roughly constant exponential term in the unsteady locus for velocity.

In the decelerating case we see that the pitch angle response is somewhat more unstable than the frozen case, due to the increasing exponential.

The velocity response initially appears somewhat more stable than the frozen case, as indicated by the unsteady locus, and ultimately is less stable.

These results indicate how in higher-order systems, the approximate approach developed may be used to estimate the effects of changing flight conditions on the dynamics of aircraft.

D. Summary

To conclude this chapter, we summarize some of the important features of the effects of varying flight velocity on the dynamics of airplanes.

The most unusual aspect of the phenomena investigated relates to the basic property of linear differential equations with time-varying coefficients; that is, different variables in a problem generally have different time histories. With respect to the short period motion of an airplane, for example, the pitching velocity and vertical velocity responses are least affected by variable velocity, and generally are influenced in a manner that might be expected. Angle of attack and normal acceleration

are more strongly affected and may appear quite different from pitching velocity and vertical velocity.

For an airplane with stability derivatives that are linearly proportional to velocity, the only influence of variable flight velocity is a distortion of the time scale when considering the variables, pitching velocity, and vertical velocity. These responses are independent of flight velocity when expressed in terms of distance traveled. This would lead us to believe that, in general, the effects on the short period motion are of a rather orderly nature, except for very unusual stability derivative variations.

The most important additional effect on the pitching and vertical velocity responses appears to be the influence of the variable frequency on the damping of the motion. The importance of this effect roughly depends upon magnitude of the fractional change in frequency per cycle compared to the quasi-steady damping ratio, when the short period frequency is proportional to other than the first power of the velocity.

For other stability derivatives, in most cases of interest, approximate methods may be used to investigate the nature of the time-varying effects, i. e., when the steady motion is relatively fast and lightly damped. The approximate methods are restricted to cases where the fractional change in the frequency per cycle is reasonably small. Physically, the approximations are restricted to reasonably high frequencies. Whenever the change in frequency with time is such that, during an "instantaneous cycle," the frequency less than doubles, we can use approximate methods. It generally appears that only for extremely low steady flight frequencies would this relationship be violated, so that any reasonably fast phenomena which would be of importance in piloting and controlling the airplane can be handled by approximate methods.

The slow motions of the airplane would be most strongly influenced by varying velocity, and these are precisely the ones of least concern in piloting the airplane.

In higher-order systems, similar effects appear to be present. Due to the additional number of parameters entering the problem, it is more difficult to generalize the results. Again, it would be expected that the approximate methods discussed here would cover the cases of most interest; that is, the cases when the steady flight motion of the airplane is reasonably rapid. Unsteady effects would become more pronounced when the motion is slow, but would be of less importance to the pilot.

These ideas are drawn from a limited investigation of certain specific cases; however, the approximate methods presented here can be used to obtain many specific conclusions quite easily.

It generally appears then, that with regard to piloting the airplane, as judged from the response of the vehicle to disturbances, there would be no particularly severe or unusual phenomena due to variable flight velocity that would not be foreseen on the basis of a careful investigation of the frozen characteristics of the vehicle and use of the approximations presented here. We are considering the dynamic motions of an aircraft during a maneuver in which the pilot would be paying careful attention to the task of flying the aircraft. Thus, in this type of situation, only rapid and unusual phenomena would be of primary concern. Problems of stability associated with the slower modes of motion of the airplane are of little concern in this situation, and thus would only be of interest in periods of long, steady flight that can be treated by conventional theory.

It is difficult to estimate at this time the importance of the differences in the different vehicle coordinates during accelerating and decelerating flight and how these various cues may affect the piloting task. The instabilities in angle of attack that may occur at low speeds are probably of little importance to the pilot. However, the apparent instabilities that may occur in normal acceleration in accelerating flight may have some effect on piloting technique. Effects such as this would tend to be most important on vehicles with light aerodynamic damping, when it is desirable to pay particular attention to the possible importance of unsteady effects.

CONCLUSIONS

1. The analysis of the dynamics of systems described by linear differential equations with variable coefficients should include consideration of all the coordinates, as well as the derivatives of the coordinates, since, in general, each of these quantities will exhibit a different amplitude variation with time.
2. For a given percentage change in the characteristic roots of a system per unit time, the departure of time-varying characteristics from frozen (constant coefficient) characteristics increases as the order of the system increases, and as the spacing between frozen roots on the complex plane decreases.
3. Frequency variation as a function of time gives rise to apparent damping.
4. The WKB approximation, generally speaking, applies to second-order systems with light damping.
5. The effect of varying flight velocity on the short period motion of an aircraft with stability derivatives linearly proportional to velocity may be summarized as follows:

The alteration of the time histories of vertical velocity and pitching velocity from frozen characteristics is a distortion of the time scale.

The apparent damping of the normal acceleration response and the angle of attack response is changed.

For an aircraft that is stable in steady flight, the angle of attack response may be unstable in decelerated flight. The normal acceleration response may be unstable in accelerated flight.

6. Other stability derivative variations with velocity produce changes in the apparent damping of the vertical velocity and pitching velocity responses.
7. For conventional fixed-wing aircraft with present horizontal acceleration capabilities, time-varying effects arising from changing flight velocity appear small at normal flight speeds.

8. Significant variations from frozen characteristics are possible at low speeds, such as encountered by VTOL aircraft, particularly in the angle of attack and normal acceleration time histories.

RECOMMENDATIONS

1. Further experimental data are necessary to ascertain the dependence of the stability derivatives of VTOL aircraft on flight condition to aid in understanding the effects of rapid transitions on the nature of the stability derivatives as functions of velocity, tilt angle, and power.
2. The effects of control motions and sustained disturbances on the response of aircraft in accelerated flight should be investigated. In most cases of interest, the approximate methods developed here can be used for investigation of forced response. The objection raised to the use of the WKBJ method in studying the forced response of systems in Reference 37 is not valid. The relationship between the impulse response and the step response of a time-variable system has not been expressed correctly.
3. Once a firm picture of the steady flight dynamics of VTOL aircraft has been established, experiments to verify these effects should be investigated.
4. Further consideration of the relative importance of the validity of the linearized time-varying approach as contrasted to the nonlinear approach, and the implications of each approach to the system are desirable.

REFERENCES

1. Hacker, T., "On the Longitudinal Stability of an Airplane in Unsteady Straight Level Flight," Revue de Mécanique Appliquée, Tome I, Number 2, 1956, pp 23-36.
2. Merrick, V. K., "The Influence of Rate of Change of Altitude and Acceleration on the Longitudinal Dynamic Stability of Aircraft and Missiles," IAS Proceedings of the National Specialists Meeting on Dynamics and Aeroelasticity, Fort Worth, Texas, November 6-7, 1958, pp 118-127.
3. Friedrich, Hans R., and Dore, Frank J., "The Dynamic Motion of a Missile Descending Through the Atmosphere," Journal of the Aeronautical Sciences, Volume 22, Number 9, September 1955, pp 628-632, 638.
4. Oswald, Telford W., "Dynamic Behavior During Accelerated Flight With Particular Application to Missile Launching," Journal of the Aeronautical Sciences, Volume 23, Number 8, August 1956, pp 781-791.
5. Laitone, E. V., "Dynamic Longitudinal Stability Equations for the Re-Entry Ballistic Missile," Journal of the Aerospace Sciences, Volume 26, Number 2, February 1959, pp 94-98.
- 5a. Coakley, T. J., Laitone, E. V., and Maas, W. L., Fundamental Analysis of Various Dynamic Stability Problems for Missiles, Series No. 176, Issue No. 1, Institute of Engineering Research, University of California, Berkeley, California, June 1961, 96 pp.
6. Garber, T. B., "On the Rotational Motion of a Body Re-Entering the Atmosphere," Journal of the Aerospace Sciences, Volume 26, Number 7, July 1959, pp 443-449.
7. Leon, Herman I., "Angle of Attack Convergence of a Spinning Missile Descending Through the Atmosphere," Journal of the Aerospace Sciences, Volume 25, Number 8, August 1958, pp 480-484.
8. Allen, H. Julian, Motion of a Ballistic Missile Angularly Misaligned With the Flight Path Upon Entering the Atmosphere and its Effect Upon Aerodynamic Heating, Aerodynamic Loads, and Miss Distance, Technical Note 4048, NACA, Washington, D. C., October 1957, 66 pp.

9. Tobak, Murray, and Allen, H. Julian, Dynamic Stability of Vehicles Traversing Ascending or Descending Paths Through the Atmosphere, Technical Note 4275, NACA, Washington, D. C., July 1958, 35 pp.
10. Reed, Wilmer H. , III, "Effects of a Time-Varying Test Environment on the Evaluation of Dynamic Stability With Application to Flutter Testing," Journal of the Aerospace Sciences, Volume 25, Number 7, July 1958, pp 435-443.
11. Collar, A. R. , "On the Stability of Accelerated Motion: Some Thoughts on Linear Differential Equations With Variable Coefficients," The Aeronautical Quarterly, Volume VIII, Part 4, November 1957, pp 309-330.
12. Graham, Dunstan, Brunelle, Eugene J., Jr., Johnson, Walter, and Passmore, Harry, III, Engineering Analysis Methods for Linear Time Varying Systems, Technical Documentary Report ASD-TDR -62-362, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, January 1963, 242 pp.
13. Malkin, I. G. , Theory of Stability of Motion, AEC-tr-3352, translated from a publication of the State Publishing House of Technical-Theoretical Literature, Moscow-Leningrad, 1952, 456 pp.
14. Seckel, Edward, Stability and Control of Airplanes and Helicopters, Academic Press, New York, 1964, 506 pp.
15. Den Hartog, J. P. , Mechanical Vibrations, Third Edition, McGraw-Hill Book Company, New York, 1947, 478 pp.
16. Ince, E. L. , Ordinary Differential Equations, Dover Publications, New York, 1956, 558 pp.
17. Davis, Harold T. , Introduction to Nonlinear Differential and Integral Equations, Dover Publications, New York, 1962, 566 pp.
18. Jahnke, Eugene, and Emde, Fritz, Tables of Functions With Formulae and Curves, Fourth Edition, Dover Publications, New York, 1945, 380 pp.
19. Smirnov, A. D. , Tables of Airy Functions and Special Confluent Hypergeometric Functions for Asymptotic Solutions of Differential Equations of the Second Order, translated from the Russian by D. G. Fry, Pergamon Press, New York, 1960, 260 pp.

20. Jeffreys, Harold, and Jeffreys, Bertha, Methods of Mathematical Physics, Third Edition, Cambridge University Press, Cambridge, England, 1956, 714 pp.
21. Brunelle, E., Transient and Nonlinear Effects on High Speed, Vibratory Thermoelastic Instability Phenomena, Part I - Theoretical Considerations, Wright Air Development Division Technical Report 60-484, Part I, July 1960.
22. Wintner, Aurel, "The Schwarzian Derivative and the Approximation Method of Brillouin," Quarterly of Applied Mathematics, Volume XVI, Number 1, April 1958, pp 82-86.
23. Bellman, Richard, Stability Theory of Differential Equations, McGraw-Hill, New York, 1953, 166 pp.
24. NASA Conference on V/STOL Aircraft, A Compilation of the Papers Presented, Langley Research Center, Langley Field, Virginia, November 17-18, 1960.
25. Newsom, William A., Jr., and Tosti, Louis P., Force-Test Investigation of the Stability and Control Characteristics of a 1/4 Scale Model of a Tilt-Wing Vertical-Take-Off-and-Landing Aircraft, Memo 11-3-58L, NASA, Washington, D. C., January 1959, 53 pp.
26. Newsom, William A., Jr., Aerodynamic Characteristics of Four-Duct Tandem VTOL-Aircraft Configurations, Technical Note D-1481, NASA, Washington, D. C., January 1963, 151 pp.
27. Zadeh, L. A., "Frequency Analysis of Variable Networks," Proc. IRE, Volume 38, 1950.
28. Dukes, Theodor A., Carballal, Jose M., and Lion, Paul M., Some Dynamic Aspects of Stability in Low-Speed Flying Machines, Technical Report 63-56, TRECOM, Fort Eustis, Virginia, November 1964, 47 pp.
29. Dingle, R. B., "The Method of Comparison Equations in the Solution of Linear Second-Order Differential Equations (Generalized W. K. B. Method)," Applied Scientific Research, Section B, Volume 5, Number 5, 1956, pp 345-367.
30. Olver, F. W. J., "Error Bounds for the Liouville-Green (or WKB) Approximation," Proceedings of the Cambridge Philosophical Society, Volume 57, Part 4, October 1961, pp 579-810.

31. Parlett, Lysle P., and Kirby, Robert H., "Test Techniques Used by NASA for Investigating Dynamic Stability Characteristics of V/STOL Models," AIAA Aerodynamic Testing Conference, Washington, D.C., March 9-10, 1964, pp 42-49.
32. Bellman, Richard, Perturbation Techniques in Mathematics, Physics, and Engineering, Holt, Rinehart, and Winston, New York, 1964, 118 pp.
33. Evans, Walter, R., Control System Dynamics, McGraw-Hill, New York, 1954, 282 pp.
34. Kulsrud, R. M., "Adiabatic Invariant of the Harmonic Oscillator," Physical Review, Volume 106, Number 2, April 15, 1957, pp 205-207.
35. Mitchell, Robert G., Full-Scale Wind-Tunnel Test of the VZ-2 VTOL Airplane With Particular Reference to the Wing Stall Phenomena, Technical Note D-2013, NASA, Washington, D.C., December 1963, 58 pp.
36. Cesari, Lamberto, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Second Edition, Springer, Berlin, 1963, 271 pp.
37. Pipes, L. A., "Four Methods for the Analysis of Time-Variable Circuits," Trans. IRE, CT-2, Number 1, March 1955.
38. Perkins, Courtland D., and Hage, Robert E., Airplane Performance Stability and Control, John Wiley and Sons, Inc., New York, 1949, 493 pp.
39. Kuhn, Richard E., and Draper, John W., Investigation of the Aerodynamic Characteristics of a Model Wing-Propeller Combination and of the Wing and Propeller Separately at Angles of Attack up to 90°, Report 1263, NACA, 1956, 40 pp.
40. Hargraves, Charles R., An Analytical Study of the Longitudinal Dynamics of a Tilt-Wing VTOL, Department of Aeronautical Engineering Report No. 561, Princeton University, June 1961, 169 pp.
41. Durand, William Frederick, Ed., Aerodynamic Theory, reprinted by Dover Publications, New York, 1963, Volume II, Division E, "General Aerodynamic Theory-Perfect Fluids," by vonKármán, Th. and Burgers, J. M.

42. Putman, W. F., Results of Experiments on a Tilt-Wing VTOL Aircraft Using the Princeton University Forward Flight Facility, Department of Aeronautical Engineering Report No. 542, Princeton University, May 1961, 223 pp.

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APPENDIX I

TRANSITION EQUATIONS

We consider a simplified approach to the prediction of the level flight transition time histories of VTOL aircraft. For comparative purposes, we investigate simplified models of a deflected jet and a tilt-wing VTOL.

In particular, we present results showing how velocity and thrust (power setting) vary with tilt angle as a function of a uniform tilt rate. The results indicate additional effects that may influence the perturbation dynamics of VTOL aircraft during a transition, but have not been taken into account in this study. It is considered that more experimental data are desirable before going into further detail than is presented here. Also, the pilot has considerable flexibility in performing this maneuver, a factor not easily taken into account in an analytical treatment.

Therefore, these results are presented to give an estimate of the magnitude of the variation in flight conditions that may be encountered at low speeds when performing a transition.

The mathematical models, although highly simplified, should indicate significant trends in this maneuver.

The problem is formulated by prescribing the time histories of the following quantities:

$i_{w_p}(t)$ tilt angle of thrust producer

$\theta_p(t)$ fuselage attitude

$w_p(t)$ vertical velocity

Equations of motion are developed, and the following quantities are determined:

$u_p(t)$ flight velocity

$\delta_p(t)$ power setting (thrust)

The pitching moment equation is not considered, and the relationship between $\theta_p(t)$ and $w_p(t)$ is selected so that transition takes place at constant attitude.

A. Deflected Jet VTOL

First, we consider a configuration with the ability to rotate engines or deflect the jet exhaust. Any aerodynamic forces induced by the inlet or exhaust flow are neglected. Only the momentum drag of the jet engine is taken into account. Induced flow effects may have significant effects on the pitching moments, but should not significantly influence the balance of forces.

From Figure 21, summation of vertical and horizontal forces yields:

$$\Sigma X = T \sin i_w - D = m\dot{u} , \quad (A-1)$$

$$\Sigma Z = W - L - T \cos i_w = 0 . \quad (A-2)$$

The solution of equations A-1 and A-2 will determine the transition time history. The transition flight path is level, and it is assumed that airplane angle of attack is held constant through a transition.

We assume that the lift can be expressed as

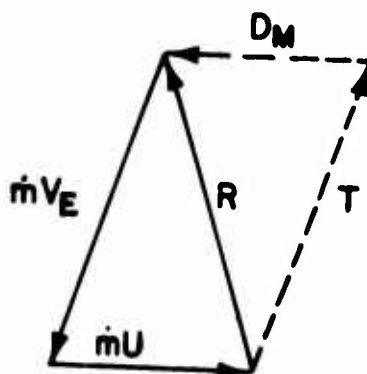
$$L = \frac{1}{2} \rho S U^2 C_L , \quad (A-3)$$

where $C_L = C_L(\alpha)$.

The drag consists of two parts: one due to the airframe, and one due to the loss of the inlet momentum of the air used by the jet engine.

$$D = D_{AC} + D_M .$$

The momentum drag, D_M , may be determined from the following diagram:



Therefore:

$$T = \dot{m} V_E ,$$

$$D_M = \dot{m} U ,$$

where

$$\dot{m} = \rho_E A_E V_E ,$$

and so

$$V_E = \sqrt{\frac{T}{\rho_E A_E V_E}} ,$$

$$D_M = \sqrt{\rho_E A_E T} U .$$

Then, the total airplane drag may be expressed as:

$$D = \frac{1}{2} \rho S U^2 C_D + \sqrt{\rho_E A_E T} U . \quad (A-4)$$

The characteristic velocity is defined as:

$$U^* = \sqrt{\frac{2W}{\rho S}} . \quad (A-5)$$

The characteristic velocity associated with the jet engine is:

$$V_E^* = \sqrt{\frac{W}{\rho_E A_E}} . \quad (A-6)$$

This is the exit velocity of the jet engine when thrust equals weight.

The characteristic time is taken as $\frac{1}{i_w}$, and the velocities in the problem are nondimensionalized by U^* .

Equations A-1 and A-2 become:

$$\left(\frac{i_w U^*}{g} \right) \frac{d\bar{U}}{di_w} = \left(\frac{T}{W} \right) \sin i_w - C_D \bar{U}^2 - \sqrt{\frac{T}{W}} \frac{\bar{U}}{\bar{V}_E^*} \quad (A-7)$$

$$1 - C_L \bar{U}^2 = \left(\frac{T}{W} \right) \cos i_w . \quad (A-8)$$

The independent variable may be considered as $i_w(t)$.

The nondimensional tilting rate is denoted by:

$$\eta = \frac{\dot{i}_w U^*}{g}.$$

This parameter is directly proportional to the average acceleration in a transition from hover to the velocity U^* .

$$\eta = \frac{\pi}{2} \tilde{a}.$$

We eliminate the thrust from equation A-7 by using equation A-8, and obtain:

$$\eta \frac{d\bar{U}}{di_w} = (1 - C_L \bar{U}^2) \tan i_w - C_D \bar{U}^2 - \frac{1}{\bar{V}_E^*} \sqrt{\frac{1 - C_L \bar{U}^2}{\cos i_w}} \bar{U}. \quad (A-9)$$

This first-order, nonlinear differential equation gives the relationship between \bar{U} and i_w during a transition. The thrust required for vertical equilibrium is determined from equation A-8. When the tilt rate is variable, η is a function of time. The variation in lift coefficient with time must be specified, and we only consider the simplest case in which the lift coefficient is constant. Steady state flight conditions are found by setting $\eta = 0$ in equation A-9, $\eta > 0$ for an accelerating transition, and $\eta < 0$ for a decelerating transition.

Typical solutions to equation A-9 for transition from hover to forward flight at various tilt rates, obtained using a digital computer, are shown in Figure 22. The airplane is assumed to have a lift-drag ratio of 10 and a jet exit velocity typical of turbojet engines. This figure clearly shows the wide variety of flight conditions that may be encountered during a transition, depending upon the tilt rate and the lift coefficient during the maneuver. By flight condition, we mean the flight velocity and thrust setting at a given tilt angle.

Admittedly, the maneuver has been treated in an idealized manner; however, the results serve to indicate what may be an important factor in the dynamics of aircraft during a rapid transition. If the stability derivatives at a given flight speed are strongly dependent upon the deflection angle and thrust setting,

then the frozen stability characteristics will be dependent upon the maneuver performed.

Since the pilot has considerable flexibility in selecting the angle of attack variation during a transition, the results presented in Figure 22 may be viewed as providing boundaries on the flight conditions encountered during a level flight transition. It is interesting to observe that if the airplane is flown through a transition at a low lift coefficient, the transition will be accomplished very quickly. Figure 23 shows the relationship between nondimensional velocity and tilt angle, for a given tilt rate, at different lift coefficients. A nondimensional velocity of 1 might be considered the end of a transition, since a lift coefficient of 1 will produce lift equal to weight. Note that at a $C_L = .16$, for example, this velocity is reached after tilting the thrust only 30° , indicating that the transition maneuver is essentially completed in a relatively short time.

There is not sufficient experimental data at the present time to consider further the nature of the aerodynamic forces and moments on these vehicles and the detailed implications of these results.

B. Tilt-Wing Aircraft

We also formulate a simplified analytical model of a tilt-wing aircraft during a transition. The main difference between the treatment of the tilt wing and the deflected jet is the assumption in the case of the tilt wing that the wing lift is proportional to slipstream dynamic pressure and acts normal to the wing chord line. This should be a satisfactory assumption for the low speed part of transition where the propeller slipstream velocity is considerably greater than the freestream velocity.

The equations of motion of this airplane resolved normal to and parallel to the thrust axis are (from Figure 21):

$$m\dot{u} \cos i_w = W \sin i_w - L_s - D_f \cos i_w, \quad (A-10)$$

$$m\dot{u} \sin i_w = T - W \cos i_w - D_f \sin i_w. \quad (A-11)$$

To compute the lift on the wing due to the slipstream, L_s , as a function of flight variables, we need additional relationships. An

approximate relationship between thrust and slipstream velocity suitable for transition is (Reference 39):

$$q_s \cong q + \frac{T}{A} . \quad (A-12)$$

We also assume that wing lift may be expressed as:

$$L_s = q_s S a_{ws} \alpha_e . \quad (A-13)$$

We consider at low speeds that only those portions of the wing that are immersed in the slipstream will produce significant aerodynamic forces. Additionally, we assume that any effects of span loading may be included in the computation of the slope of the lift curve, a_{ws} , and that a_{ws} is constant throughout the transition. This last assumption is made for simplicity. Actually, it would be expected that the lift curve slope based on slipstream dynamic pressure would be reduced in low-speed flight due to the finite extent of the slipstream compared to its value in a uniform flow (Reference 41, page 236). The effective angle of attack of the wing is taken as the angle between the wing zero lift line and the vector sum of the propeller induced velocity far downstream and the freestream velocity as shown in Figure 21 (Reference 40).

Assuming that α_e is a small angle, the law of sines yields:

$$\alpha_e = \sqrt{\frac{q}{q_s}} \cos i_w = \frac{V}{V_s} \cos i_w . \quad (A-14)$$

The resulting expression for wing lift (from A-12, A-13, and A-14) is:

$$L_s = S a_{ws} (q + \frac{T}{A}) \sqrt{\frac{q}{q_s}} \cos i_w . \quad (A-15)$$

Where for simplicity we neglect fuselage drag as not important at low flight speeds, equations A-10 and A-11 may be written as:

$$\frac{1}{g} \dot{u} = \tan i_w - \frac{S a_{ws}}{W} (q + \frac{T}{A}) \sqrt{\frac{q}{q_s}} , \quad (A-16)$$

$$\frac{1}{g} \dot{u} \sin i_w = \frac{T}{W} - \cos i_w . \quad (A-17)$$

Then we may eliminate the thrust between equations A-16 and A-17, resulting in the following first-order nonlinear differential equation for the velocity:

$$\eta' \frac{dU'}{di_w} = \tan i_w + \frac{U'^2}{2} \sin i_w - U' \sqrt{\frac{1}{\cos i_w} + U'^2 (R^2 + \frac{1}{4} \sin^2 i_w)} , \quad (A-18)$$

where

$$\eta' = \eta r$$

$$U' = \frac{\bar{U}}{r}$$

$$R = \frac{A}{S a_{ws}}$$

$$r = \sqrt{\frac{A}{S}} \frac{1}{a_{ws}} ,$$

where η and \bar{U} are defined in the section on the deflected jet aircraft.

The differential equation A-18 was solved numerically on a digital computer for values of the nondimensional parameters typical of tilt-wing aircraft. The results for one specific case are presented in Figure 24.

We note the typical behavior of these curves. For an accelerating transition, the freestream velocity is always lower and the thrust higher than in steady flight at the same tilt angle. For a decelerating transition, the flight velocity is always higher and the thrust lower than the corresponding steady flight case.

The variation in flight conditions encountered does not appear to be so wide as that encountered by the tilt jet vehicle, but is of such a magnitude that the variation in stability derivatives with the transition flight path may be significant. The decelerating transition is probably the most significant in this regard, due to the tendency to encounter higher flight velocities and lower thrust settings at high wing angles. Inspection of equations A-12 and A-14 indicates that both of these tendencies lead to higher wing angles of attack and thus to increased areas of separation over the wing. Presently available data indicate that the lateral stability characteristics are more sensitive to this phenomena

than the longitudinal characteristics (Reference 35, Reference 25, Reference 31).

For a level flight transition, the jet vehicle has greater flexibility during a transition, as a result of control of airplane angle of attack. Variation in angle of attack at low speeds with the tilt-wing vehicle should not have a particularly strong influence on the aerodynamic forces at low speeds, since the major wing forces are produced by the slipstream velocity. The effective "drag" of the tilt-wing vehicle is very high at low speeds, since the wing lift, being dependent upon slipstream velocity, acts approximately normal to the propeller thrust line, rather than normal to the freestream velocity, resulting in a considerably slower transition. Compare the shape of the acceleration time histories of Figure 22 with Figure 24. Figure 23 shows a direct comparison between a tilt-wing transition and the deflected jet transitions at the same tilt rate.

At the high-speed end of the tilt-wing transition, as the free-stream velocity becomes larger than the propeller induced velocity, the pilot does have the flexibility to vary the angle of attack of the airplane and produce significant changes in the aerodynamic forces. We do not consider those effects here, and thus our simplified model only applies for perhaps 60° of wing tilt.

Decelerating transitions are also shown for the tilt wing in Figure 24. Note that in this continuous maneuver the airplane is still at some forward speed when the wing reaches vertical. It is probable that towards the end of the decelerating maneuver, the pilot would adjust the wing rate so as to arrive at zero velocity with the wing vertical.

APPENDIX II

CONVENTIONAL AIRPLANE EQUATIONS OF MOTION IN UNSTEADY FLIGHT

We now determine the conventional airplane equations of motion for the general case of varying flight velocity. Considerations relating to the linearization of these equations are discussed in Chapter V. For completeness, we formulate the equations in both wind and body axes, to show the equivalence of the two approaches.

First we consider the force equations in the plane of symmetry, expanding the aerodynamic forces in a Taylor series, retaining only first-order terms (Reference 14, Chapter XI), and then discuss any changes that arise due to variable flight velocity.

With respect to an axis fixed to the wind, the inertia and gravity terms are (Reference 38, page 383f):

$$\begin{aligned} X_{wi} &= -m\dot{V} \\ Z_{wi} &= +mV\dot{\gamma} \\ X_{wg} &= -W \sin \gamma \\ Z_{wg} &= +W \cos \gamma \end{aligned} \tag{B-1}$$

Summation of forces normal and parallel to the wind, from Figure 21, yields:

$$Z_{wa} = -L - T \cos(i_w - A), \tag{B-2}$$

$$X_{wa} = -D + T \sin(i_w - A). \tag{B-3}$$

Expanding the vertical aerodynamic force (equation B-2), retaining only zeroth and first-order terms,

$$Z_{wp} + \Delta Z_w = -L_p - T_p \cos(i_w - A_p) - \Delta L - T_p \sin(i_w - A_p) \Delta \alpha - \Delta T \cos(i_{wp} - A_p) \tag{B-4}$$

where the subscript p refers to the predetermined path and $\Delta A = \Delta \alpha$.

We separate equation B-4 into zeroth-order and first-order terms, the zeroth-order terms determining the path dynamics and the first-order terms the perturbation dynamics.

The path forces are:

$$Z_{w_p} = -L_p - T_p \cos(i_w - A_p). \quad (B-5)$$

The perturbation forces are:

$$\Delta Z_w = -\Delta L - T_p \sin(i_w - A_p) \Delta \alpha - \Delta T \cos(i_w - A_p). \quad (B-6)$$

Similarly, we obtain two parts to the horizontal force. Linearizing the inertia and gravity terms from equation B-1, with $\dot{\gamma}_p = 0$, and then equating zeroth-order and first-order terms on each side, we obtain the wind axis equations.

a. The path equations are:

$$m\dot{V}_p + W \sin \gamma_p = -D_p + T_p \sin(i_{w_p} - A_p), \quad (B-7)$$

$$-W \cos \gamma_p = -L_p - T_p \cos(i_{w_p} - A_p). \quad (B-8)$$

b. The perturbation equations are:

$$m\dot{\Delta V} + W \cos \gamma_p \Delta \gamma = -\Delta D - T_p \cos(i_{w_p} - A_p) \Delta \alpha + \Delta T \sin(i_{w_p} - A_p), \quad (B-9)$$

$$-mV_p \dot{\Delta \gamma} + W \sin \gamma_p \Delta \gamma = -\Delta L - T_p \sin(i_{w_p} - A_p) \Delta \alpha - \Delta T \cos(i_{w_p} - A_p). \quad (B-10)$$

We now develop the body axis equations. Resolving the aerodynamic forces along the body axes, from Figure 21, we obtain:

$$X_B = L \sin A - D \cos A + T \sin i_w, \quad (B-11)$$

$$Z_B = -L \cos A - D \sin A - T \cos i_w. \quad (B-12)$$

Expanding equations B-11 and B-12, and separating zeroth- and first-order terms, the path forces are:

$$X_{B_p} = L_p \sin A_p - D_p \cos A_p + T_p \sin i_{w_p}, \quad (B-13)$$

$$Z_{B_p} = -L_p \cos A_p - D_p \sin A_p - T_p \cos i_{w_p}, \quad (B-14)$$

and the perturbation forces are:

$$\Delta X_B = \Delta L \sin A_p - \Delta D \cos A_p + (D_p \sin A_p + L_p \cos A_p) \Delta \alpha + \Delta T \sin i_{w_p}, \quad (B-15)$$

$$\Delta Z_B = -\Delta L \cos A_p - \Delta D \sin A_p - \Delta \alpha (D_p \cos A_p - L_p \sin A_p) - \Delta T \cos i_{w_p}. \quad (B-16)$$

Using the linearized gravity and inertia terms, as given in Chapter V, we obtain the body axis equations.

c. The path equations are:

$$m \dot{U}_p + W \sin \theta_p = L_p \sin A_p - D_p \cos A_p + T_p \sin i_{w_p}, \quad (B-17)$$

$$m \dot{W}_p - W \cos \theta_p = -L_p \cos A_p - D_p \sin A_p - T_p \cos i_{w_p}. \quad (B-18)$$

d. The perturbation equations are:

$$-\Delta L \cos A_p - \Delta D \sin A_p - \Delta \alpha \{D_p \cos A_p - L_p \sin A_p\} - W \sin \theta_p \Delta \theta = m(\Delta \dot{w} - U_p \dot{\theta}), \quad (B-19)$$

$$\Delta L \sin A_p - \Delta D \cos A_p + (D_p \sin A_p + L_p \cos A_p) \Delta \alpha + \Delta T \sin i_{w_p} - W \cos \theta_p \Delta \theta = m(\Delta \dot{u}). \quad (B-20)$$

In studying the short period motion, i. e., the motion with no velocity perturbation, we use only the vertical force equation. Now, we can show that equations B-10 and B-19 will be identical when there is no horizontal velocity perturbation if we apply the same constraint to both equations. For example, if we apply the condition of prescribed velocity along the X body axis, the constraint relationships are:

$$\Delta \dot{u} = \Delta \dot{V} \cos A_p - (\dot{V}_p \Delta \alpha + V_p \dot{\Delta \alpha}) \sin A_p = 0 \quad (B-21)$$

and

$$\Delta X_{B_a} = -W_p \dot{\theta}_p.$$

Equations B-10 and B-19 will be different if the condition $\Delta \dot{V} = 0$ is applied to the wind axis equation and the condition $\Delta \dot{u} = 0$ applied to the body axis equation, if the axes are not aligned ($A_p \neq 0$).

If the body axis is aligned with the wind ($A_p \cong 0$), then the body axis equations (B-19 and B-20) become:

$$\Delta L + \Delta \alpha \left\{ \underline{\underline{D_p}} - m \dot{U}_p + T_p \sin i_{wp} - W \sin \theta_p \right\} + W \sin \theta_p \Delta \theta = -m(\Delta \dot{w} - U_p \dot{\theta}), \quad (B-22)$$

$$- \Delta D - W \cos \theta_p \Delta \theta + \Delta \alpha \left\{ \underline{\underline{L_p - W \cos \theta_p}} - T_p \cos i_{wp} \right\} = m \Delta \dot{u}; \quad (B-23)$$

and the wind axis equations (B-9 and B-10) become:

$$\Delta L + \Delta \alpha \left\{ \underline{\underline{T_p \sin i_{wp}}} + m \dot{V}_p + D_p + W \sin \gamma_p \right\} + W \sin \gamma_p \Delta \gamma = m V_p \dot{\gamma}, \quad (B-24)$$

$$- \Delta D - W \cos \gamma_p \Delta \gamma - \left\{ \underline{\underline{T_p \cos i_{wp}}} - L_p + W \cos \gamma_p \right\} \Delta \alpha = m \Delta \dot{V}, \quad (B-25)$$

where for simplicity $\Delta T = 0$.

The important thing to note is how the appearance of the equations may change, depending upon whether or not the path relationships are used to express some of the terms in an alternate form. For convenience, these alternate forms are given in parentheses (equations B-7, B-8, B-17, and B-18). The form of these equations, using the underlined terms, indicates that there will be no direct effects of varying flight velocity in the equations of motion. For example, if the thrust of the airplane is varied to accelerate the airplane, the only way it would appear in equations B-22 and B-23 would be through power effects on the lift and drag. When the axes are initially aligned to the first order, the vertical force equations B-22 and B-24 are identical when $\Delta \dot{u} = \Delta \dot{V} = 0$, and so the same short period equations will be obtained with either axis system.

In this form the drag equations are identical and this equivalence, in accelerated flight, can be shown by recalling that:

$$\Delta w = \Delta \alpha V_p$$

and

$$\Delta \dot{w} = \Delta \dot{\alpha} V_p + \Delta \alpha \dot{V}_p,$$

by using the path relationship, equation B-8.

References 3, 5, and 8 use wind axes and consider a missile without thrust. Laitone, in Reference 5, comparing equation B-24, derived for the case of zero thrust, to equation B-22, considers as an example that the drag of the vehicle is large and the acceleration is small. This is not physically possible, because the path equations are not satisfied and therefore the equations do not agree. We will use a body axis system and assume that the path motion is near level flight, such that $\cos \theta_p \approx 1$ and $\sin \theta_p \Delta \gamma$ is second order.

We assume, for this conventional case, that the lift and drag are not dependent upon power, and may be expressed as:

$$L = \frac{1}{2} \rho S U^2 C_L, \quad (B-26)$$

$$D = \frac{1}{2} \rho S U^2 \left(C_{D_0} + \frac{C_L^2}{\pi AR e} \right). \quad (B-27)$$

Expanding these expressions in a Taylor series and placing them into equations B-22 and B-23, we obtain equations B-28 and B-29. The pitching moment equation may be written as equation B-30. The assumption that there are no significant power effects results in $C_{m_u} = 0$, and for simplicity we have taken $C_{m_{\dot{w}}} = 0$ (Reference 14, Chapter XI). It is physically clear that there are no direct effects of acceleration on the pitching moment equation, since the origin of the axis system has been taken at the center of gravity of the vehicle.

The differential equations describing the motion of an airplane with no power effects are (we drop the Δ 's with the understanding that any quantity without a subscript p is a perturbation quantity):

$$-\frac{du}{dt} - g\theta = \frac{1}{\mu c} \left\{ (U_p C_{D_p}) u + U_p C_{L_p} \left(1 - \frac{C_L \alpha}{\pi AR e} \right) w \right\}, \quad (B-28)$$

$$-\frac{dw}{dt} + U_p q = \frac{1}{\mu c} \left\{ (U_p C_{L_p}) u + \frac{U_p}{2} (C_{L_\alpha} + C_{D_p}) w \right\}, \quad (B-29)$$

$$\frac{dq}{dt} = \frac{1}{2\mu k_y^2} \left\{ C_{m_\alpha} U_p w + C_{m_q} c U_p q \right\}, \quad (B-30)$$

$$\frac{d\theta}{dt} = q. \quad (B-31)$$

These equations are identical in appearance to the equations of motion in steady flight (Reference 14, Chapter XI). In general, U_p , C_{D_p} , and C_{L_p} are functions of time.

Since the flight velocity U_p is a function of time, we denote the non-dimensional time as x , and relate it to real time in the following fashion:

$$x = \int_0^t U_p(s) ds. \quad (B-32)$$

When U_p is constant, equation B-32 differs from the classical definition of aerodynamic time (Reference 14, page 187) by the factor (μc) . x is the distance traveled by the airplane.

We consider

$$q(t) = q(x),$$

$$u(t) = u(x),$$

$$w(t) = w(x),$$

and apply transformation B-32 to equations B-28 through B-30 to obtain:

$$-\frac{du}{dx} - \frac{g}{U_p} \int_0^x \frac{q ds}{U_p} = \frac{1}{\mu c} \left\{ C_{D_p} u + C_{L_p} \left(1 - \frac{C_{L_\alpha}}{\pi AR e} \right) w \right\}, \quad (B-33)$$

$$-\frac{dw}{dx} + q = \frac{1}{\mu c} \left\{ C_{L_p} u + \frac{1}{2} (C_{L_\alpha} + C_{D_p}) w \right\}, \quad (B-34)$$

$$\frac{dq}{dx} = \frac{1}{2\mu k_y^2} \left\{ C_{m_\alpha} w + C_{m_q} c q \right\}. \quad (B-35)$$

Only the appearance of the gravity term (the second term on the left-hand side of equation B-33) prevents the exact integration of these equations for flight at constant lift coefficient. This case is artificial for the conventional airplane, but we may consider these equations as a first-order approximation to a tilt jet vehicle at low speeds when the jet engine provides no appreciable contributions to the stability derivatives. In the following, due to the algebra involved, it is convenient first to determine the differential equations describing each variable considering the independent variable as x and then to transform the independent variable back to time. We consider briefly some aspects of the equations of motion when the flight velocity is changing.

A. Short Period Motion

The short period equations are obtained immediately upon dropping the terms involving the perturbation velocity, u , and retaining only the vertical force and moment equations (Reference 14).

$$\frac{dw}{dx} - q = \frac{1}{\mu c} \left\{ \frac{1}{2} (C_{L_\alpha} + C_D) w \right\}, \quad (\text{B-36})$$

$$\frac{dq}{dx} = \frac{1}{2\mu k_y} \left\{ C_{m_\alpha} w + C_{m_q} c q \right\}. \quad (\text{B-37})$$

These equations and the interpretation of their solution are discussed at length in Chapter V. Equations B-36 and B-37 are constant coefficient equations and therefore the differential equations for pitching velocity and vertical velocity are identical and are:

$$\begin{aligned} \frac{d^2 w}{dx^2} + \left[\frac{1}{2\mu c} \left\{ C_{L_\alpha} + C_D - C_{m_q} \left(\frac{c}{k_y} \right)^2 \right\} \right] \frac{dw}{dx} + \\ \left[- \frac{C_{m_\alpha}}{2\mu k_y} - \frac{1}{4\mu^2 k_y^2} (C_{L_\alpha} + C_D) C_{m_q} \right] w = 0. \end{aligned} \quad (\text{B-38})$$

Using the relationship, B-32, we transform equation B-38 into the time domain:

$$\frac{d^2 w}{dt^2} + \left[2\sigma_B U_p - \frac{\dot{U}}{U_p} \right] \frac{dw}{dt} + \left[\omega_B^2 U_p^2 \right] w = 0. \quad (\text{B-39})$$

The equations for pitching velocity and vertical velocity are identical in the time domain although this is not necessarily true in a coupled system with variable coefficients (see Chapter III). It is interesting to note that although there was no difference in the equations of motion due to unsteady flight, there is a difference in the differential equations describing pitching velocity and vertical velocity, due to the appearance of the term \dot{U}_p / U_p in equation B-39.

Now, let us consider the equation for angle of attack. Using the relationship between α and w , we obtain for $\alpha(t)$:

$$\ddot{\alpha} + \left[2\sigma_B U_p + \frac{\dot{U}_p}{U_p} \right] \dot{\alpha} + \left[\frac{\ddot{U}_p}{U_p} - \left(\frac{\dot{U}_p}{U_p} \right)^2 + 2\sigma_B U_p + \omega_B^2 U_p^2 \right] \alpha = 0. \quad (B-40)$$

Equation B-40 differs from equation B-39. The reason for this is clear if we examine the relationship between vertical velocity and angle of attack given above. The angle of attack as a function of time will appear quite different than vertical velocity, since U_p is a function of time.

The differential equation for normal acceleration is obtained from the relationship

$$N_z = \frac{1}{2\mu c} (C_{L\alpha} + C_D) U_p w$$

$$\frac{d^2 N_z}{dt^2} + \left(2\sigma_B U_p - 3\frac{\dot{U}_p}{U_p} \right) \frac{dN_z}{dt} + \left(\omega_B^2 U_p^2 - \frac{\ddot{U}_p}{U_p} - \dot{U}_p \left[2\sigma_B - 3\frac{\dot{U}_p}{U_p^2} \right] \right) N_z = 0. \quad (B-41)$$

Again, equation B-41 differs from equation B-39 when U_p is a function of time.

B. Phugoid Motion

We briefly consider the long period motion of an airplane. When the airplane has a high level of attitude stability, we may approximate the long period motion of an airplane by neglecting the perturbation in vertical velocity and retain only the force equations (equations B-28 and B-29).

$$-\frac{du}{dt} - g\theta = \frac{1}{\mu c} U_p C_{D_p} u , \quad (B-42)$$

$$+ q = \frac{1}{\mu c} C_{L_p} u . \quad (B-43)$$

It is interesting to note that if the drag term is neglected, then the velocity does not appear in these equations, and the nature of the motion will depend upon how C_{L_p} varies with time. The equations for the two variables are:

$$\frac{d^2u}{dt^2} + \frac{g}{\mu c} C_{L_p} u = 0 , \quad (B-44)$$

$$\frac{d^2\psi}{dt^2} + \frac{g}{\mu c} C_{L_p} \psi = 0 , \quad (B-45)$$

where

$$\psi = \int_0^t \theta(s) ds . \quad (B-46)$$

Note that if we consider flight at a constant lift coefficient, then equations B-44 and B-45 are constant coefficient equations.

For flight at constant lift coefficient with varying velocity, the phugoid and short period equations can be exactly integrated but most easily in different coordinate systems.

The short period equations are constant coefficient equations when the independent variable is distance traveled; i. e., the wave length of the short period is independent of flight speed.

C. Three-Degree-of-Freedom Motion

We now consider briefly the differential equation for pitching velocity for the complete three-degree-of-freedom motion. Upon eliminating variables between equations B-28 through B-31, we obtain the differential equation describing q (C_L constant):

$$\ddot{q} + [AU - 3\frac{\dot{U}}{U}] \dot{q} + [3(\frac{\dot{U}^2}{U}) + \frac{\ddot{U}}{U} - AU + BU^2] \dot{q} + CU^3 q + DU^2 \int_0^t q ds = 0, \quad (B-47)$$

where AU , BU^2 , CU^3 , DU^2 are the coefficients of the characteristic equation in steady flight.

The deceptive nature of quasi-steady terms may be noted from the fact that when C and D are equal to zero, we may integrate the remaining three terms once to obtain the equation for pitching velocity in the form of equation B-39.

$$\dot{q} + [AU - \frac{\dot{U}}{U}] \dot{q} + BU^2 q = 0.$$

The additional terms appearing in equation B-47 arise due to the fact that upon dropping the terms involving C and D , an equation in terms of pitching acceleration results, which is, in general, different from the equation for pitching velocity (see Chapter III). Equation B-47 may be written as:

$$\frac{d}{dt} [\dot{q} + (AU - \frac{\dot{U}}{U}) \dot{q} + BU^2 q] = -CU^3 q - DU^2 \int_0^t q ds, \quad (B-48)$$

or as

$$\ddot{q} + (AU - \frac{\dot{U}}{U}) \dot{q} + BU^2 q = -C \int_0^t U^3 q ds - D \int_0^t \theta(s) U^2 ds. \quad (B-49)$$

Thus, we are assuming that the attitude change is small and that its integral is small when we make the short period approximation. Roughly, it would appear that the short period approximation is satisfactory as long as the motion is of reasonably high frequency such that the terms on the left-hand side dominate. A and B are typically much larger than C and D .

D. Downwash Lag

To conclude this section, we briefly consider the downwash lag term, since certain peculiarities arise in unsteady flight. The downwash lag derivative, $M_{\dot{w}}$, is usually reflected as an increase in the damping of the short period motion. It should also be recalled that the term $M_{\dot{w}}$ is an approximation to a nonlinear effect, accounting for the lag in wing downwash arriving at the tail. If we wish to consider this term at very low flight speeds

where the time delay is large, it may be necessary to take the nonlinearity into account to properly represent this effect.

In the formulation of this term in accelerated flight, note that the downwash velocity is delayed, and not the downwash angle, so that the effect would be approximately represented by (Reference 14):

$$\Delta M_T = \frac{\partial M}{\partial \alpha_T} \cdot \left(\frac{w(t - \tau)}{U(t)} \right) = \frac{\partial M}{\partial \alpha_T} \frac{1}{U(t)} \left[w(t) + \frac{dw}{dt} (-\tau) \right].$$

Therefore,

$$M_{\dot{w}} = \frac{\partial M}{\partial \alpha_T} \left(-\frac{\tau}{U(t)} \right),$$

where τ is the time taken for a downwash change at the wing to reach the tail.

The presence of varying flight velocity will affect the relationship between the vertical velocity derivatives and the angle of attack derivatives; i. e.,

$$\Delta M(w, \dot{w}) = \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial \dot{w}} \Delta \dot{w},$$

$$\Delta M(\alpha, \dot{\alpha}) = \frac{\partial M}{\partial \alpha} \Delta \alpha + \frac{\partial M}{\partial \dot{\alpha}} \Delta \dot{\alpha},$$

and as we have seen above, $\frac{\partial M}{\partial \dot{w}}$ is directly computed.

Since $\alpha = \frac{w}{U_p}$,

$$\dot{\alpha} = \frac{\dot{w}}{U_p} - \frac{w}{U_p^2} \dot{U}_p.$$

We have

$$\frac{\partial M}{\partial w} = \left[\frac{1}{U_p} \frac{\partial M}{\partial \alpha} - \frac{\dot{U}_p}{U_p^2} \frac{\partial M}{\partial \dot{\alpha}} \right],$$

$$\frac{\partial M}{\partial \dot{w}} = \frac{1}{U_p} \frac{\partial M}{\partial \dot{\alpha}},$$

for the relationships between derivatives when the downwash lag is included.

APPENDIX III

STABILITY DERIVATIVES OF VTOL AIRCRAFT

Here we discuss some general features of the aerodynamic forces and moments acting on a VTOL aircraft at low speeds, and how they influence the analysis presented. Specific comments are restricted to tilt-wing aircraft; however, much of the discussion is applicable to propeller driven VTOL aircraft in general.

There are two aspects of the flight of VTOL aircraft at low speeds which are distinctly different from the conventional airplane that relate to this investigation. The first is the fact that the configuration of the airplane is variable, so that, in particular, the moment arms of the wing and propeller forces vary with the tilt angle of the wing. The second is the fact that the flight condition of the airplane, and thus the aerodynamic forces acting on the vehicle, are dependent upon the time history of the transition motion of the vehicle. The conventional airplane in level flight possesses a single relationship between angle of attack and flight velocity, while the VTOL aircraft, obtaining lift from the thrust of the propellers, has a relationship between wing tilt angle and velocity that depends upon whether the motion is steady flight or a flight condition encountered during a transition. This particular problem is discussed in Appendix I.

We define the flight condition of a VTOL aircraft by the following quantities (in level flight): the flight speed, fuselage attitude, wing tilt angle, power setting (e. g. , propeller pitch), and longitudinal control setting. We consider only motions in the plane of symmetry, and, for simplicity, do not include additional features that may be present, such as flap setting and programmed tail incidence, or variable propeller rotational speed.

Then, the flight condition, and therefore the aerodynamic forces and moments and the stability derivatives will be determined by the instantaneous values (no unsteady effects are included) of the flight speed, wing tilt angle, power setting, and longitudinal control setting. The particular relationship that exists among these four quantities at any instant of time depends upon the time history of the motion. That is, at a given wing tilt angle, the flight speed, power setting, and longitudinal control setting will depend upon whether the vehicle is in steady flight, accelerating or decelerating, and so will the stability derivatives.

Let us examine some of these effects in detail, particularly in relation to the linearity of the derivatives. For simplicity we neglect the influence of the longitudinal control setting (this may be an important effect, with respect to the influence of elevator setting on velocity stability), so that the stability derivatives are then functions of three quantities: flight speed, wing incidence, and power setting. Then, the change in a particular stability derivative in going from one flight condition to a neighboring one may be expressed in a Taylor series as:

$$x_{\xi_2} = x_{\xi_1} + \frac{\partial x_{\xi}}{\partial u} \Delta u + \frac{\partial x_{\xi}}{\partial i_w} \Delta i_w + \frac{\partial x_{\xi}}{\partial \delta_p} \Delta \delta_p \dots \quad (C-1)$$

We wish to consider which of these changes are linear, or external effects, and which are nonlinear, as discussed in Chapter V.

The second term on the right-hand side is a nonlinear change. The third term includes both nonlinear effects and linear effects. Since a change in wing incidence is equivalent to a change in wing angle of attack, variations of this nature associated with the wing-propeller combination are nonlinearities. Contributions to the change in a stability derivative because of changing moment arms or a variation in the downwash field at the tail are linear effects. The last term, due to a change in power setting, is purely a linear or external change of the stability derivative and does not influence the linearity of the description.

Thus the stability derivatives will change with flight condition. These changes may be either due to nonlinear effects or linear (external) effects, and in a specific case, the various sources should be investigated to determine the validity of a linearized description of the perturbation motion.

At the present time there is a lack of experimental data necessary to give a complete and detailed description of the variation of the stability derivatives in the manner indicated by equation C-1.

The trends of the important stability derivatives determining the initial response of the vehicle, the damping in pitch, the attitude stability, and the variation of vertical force with angle of attack of the airplane are reasonably clear.

Let us consider the damping in pitch and the attitude stability, and then other derivatives that may be important. The emphasis of the description is to point out the general features of the stability derivatives without resorting to detailed calculations.

A. Variation in Vertical Force With Angle of Attack (Z_α)

In addition to the dependence of this derivative on the flight condition, there will be a direct contribution in accelerating flight, since the forces in the X direction are no longer balanced.

First, consider a conventional airplane, with thrust, T , a body-fixed force, and drag, D , parallel to the wind by definition. Then the variation in vertical force with angle of attack is:

$$\frac{\partial Z}{\partial \alpha} = - \left(\frac{\partial L}{\partial \alpha} + D \right), \quad (C-2)$$

and the horizontal equilibrium equation is:

$$T_p - D_p = m \dot{U}. \quad (C-3)$$

An alternate form of equation C-2 is obtained by using equation C-3:

$$\frac{\partial Z}{\partial \alpha} = - \left(\frac{\partial L}{\partial \alpha} + T_p - m \dot{U} \right). \quad (C-4)$$

Since lift varies linearly with angle of attack below stall, the lift curve slope of the airplane in steady level flight is:

$$\frac{\partial L}{\partial \alpha} = \frac{W}{\alpha - \alpha_{oL}}. \quad (C-5)$$

Therefore,

$$\frac{\partial Z}{\partial \alpha} = - W \left(\frac{1}{\alpha - \alpha_{oL}} + \frac{1}{\left(\frac{L}{D}\right)} \right), \quad (C-6)$$

where, usually,

$$\frac{1}{\alpha - \alpha_{oL}} \gg \frac{1}{\left(\frac{L}{D}\right)}. \quad (C-7)$$

Given the flight velocity in level flight, the angle of attack is determined, and the rate of change of vertical force with angle of attack is not dependent upon acceleration, unless the airplane is decelerated by increasing drag.

Now, for a tilt-wing airplane or other VTOL at slow flight speeds, a similar expression may be developed. However, due to the

complicated aerodynamics of the vehicle we do not distinguish between thrust and drag.

The majority of the vertical aerodynamic force on a tilt wing is developed by the propeller-wing combination with a well defined angle of zero lift, when there is no flap deflection. Experimental data indicates that the variation of the lift force with angle of attack is reasonably linear when propeller blade angle and rotational speed are maintained constant (References 25 and 35). This result should not be confused with one method of wind tunnel testing in which propeller thrust is held constant as the angle of attack is varied (Reference 39) when there is the appearance of stall. Assuming that the variation of the total vertical force perpendicular to the wind is linear, then

$$\frac{\partial \mathcal{L}}{\partial \alpha} \cong \frac{W}{\tan(i_w - \alpha_{OL})} \quad (C-8)$$

in steady, level flight. Equation C-8 includes the thrust force. The tangent is used, since this quantity approaches zero as the flight velocity approaches zero ($i_w \rightarrow 90^\circ$). This relationship shows good agreement with experimental data, as shown in Figure 25.

The same relationship would be expected to be a good approximation on any other vehicle with tilting components and a well defined angle of zero lift.

Experimental measurements on a ducted fan configuration also agree with equation C-8 at low forward speeds where there are no significant aerodynamic forces from the fixed wing (Reference 26).

Now, the rate of change of vertical force with respect to angle of attack includes the effect of an unbalanced horizontal force. Let X denote the net horizontal force parallel to the wind. The horizontal equilibrium relationship for level flight is:

$$X = -m \dot{U} \quad (C-9)$$

and

$$\frac{\partial Z}{\partial \alpha} = -\left(\frac{\partial \mathcal{L}}{\partial \alpha} + X\right) \quad (C-10)$$

Therefore:

$$\frac{\partial Z}{\partial \alpha} \cong -W \left(\frac{1}{\tan(i_w - \alpha_{o_L})} - \frac{\dot{U}}{g} \right). \quad (C-11)$$

Now at low speeds the horizontal acceleration may have an appreciable effect on this term, as $\partial Z / \partial \alpha$ is small. In hovering, this approximate form is exact and $\partial Z / \partial \alpha$ is equal to zero.

Thus, we have the derivative in steady flight,

$$\frac{\partial Z}{\partial \alpha} = - \frac{W}{\tan(i_w - \alpha_{o_L})}, \quad (C-12)$$

modified by the horizontal acceleration term, $-\frac{\dot{U}}{g}$.

Agreement between equation C-11 and experimental data is shown in Figure 25. Available experimental data indicate that areas of separated flow on the wing do not particularly influence this derivative, such that, tentatively, no unusual behavior of this derivative is expected during rapid deceleration (References 25 and 35).

B. Damping in Pitch (M_q)

Very little experimental information is available on the damping in pitch of VTOL aircraft. Unpublished data taken on the Princeton Dynamic Model Track indicate that for a high disc loading, conventional propeller tilt-wing aircraft in hovering, the damping in pitch, for practical purposes, is negligible. Neither the wing, located close to the center of gravity, nor the highly loaded propellers provide any significant contribution. Then, the important contribution to this derivative would arise due to the increase in free stream dynamic pressure at the horizontal tail. Thus, we assume that this derivative will increase approximately linearly with flight velocity from a value of zero in hovering and is independent of power setting and tilt angle.

With low disc loading machines, or with tandem configurations, this may not be a reasonable assumption. However, at the present time there is no further experimental information available.

C. The Attitude Stability (M_w)

At very low speeds the attitude stability is due to the interactions of the wing and propeller forces on a tilt-wing configuration, as

well as the relative location of these components with respect to the center of gravity.

The general trend for vehicles of this type is shown in Reference 24, page 33. At very low flight speeds the aircraft tends to be statically unstable with attitude, and as the speed increases, the increasing dynamic pressure over the tail provides increasing attitude stability.

In general, this derivative is a complicated function of the wing tilt angle, the flight velocity, and the power setting, and therefore would depend upon transition rate. However, experimental data from Reference 25 for a tilt-wing aircraft with a high horizontal tail indicate that the attitude stability is primarily a function of flight velocity, as shown in Figure 26. If the horizontal tail is located out of the downwash field of the wing-propeller combination, the contribution of the horizontal tail would be expected to vary in this fashion. A limiting case would be no contribution from the wing-propeller, representing the conventional airplane.

Another limiting case would be the horizontal tail completely immersed in the wing-propeller slipstream. Theoretical calculations indicate that the slipstream dynamic pressure is roughly constant during a large part of the transition (Reference 40), and therefore, to the first order, the attitude stability would be constant. This is a rough approximation, but gives an indication of another possible variation in this derivative.

These three variations, the conventional airplane, an approximation to the curve shown in Figure 26, and a constant value, are investigated in Chapter V.

Further experimental data are required on the nature of the downwash field at the tail and the wing propeller forces before more accurate estimates can be made. However, it is considered that the approximations discussed, adequately represent the important trends.

Therefore, a reasonable approximation to the variation of the stability derivatives affecting the short period motion of VTOL aircraft is to assume that they are only functions of flight velocity.

Other important derivatives necessary to predict the complete longitudinal motion are the velocity stability (M_u), the rate of change of horizontal force with velocity (X_u), the rate of change of vertical force with velocity (Z_u), and the rate of change of horizontal force with vertical velocity (X_w).

These derivatives are more complicated functions of the flight conditions than those previously discussed.

The velocity stability depends, at low speeds, to a large extent, on the moment of the wing forces about the center of gravity. This dependence arises from the fact that at high wing angles a horizontal speed perturbation causes a change in wing effective angle of attack (Reference 40).

The moment of the wing forces is a function of the tilt angle and so this derivative has a relatively strong dependence on tilt angle, decreasing as the tilt angle is reduced. If we consider the wing and propeller forces only, the velocity stability is reasonably independent of speed and primarily a function of tilt angle. A possible exception is a rapid deceleration when the wing may stall, with resulting unusual behavior. The velocity stability also depends upon the horizontal tail lift coefficient, when the horizontal tail is not providing pitching moment trim, as well as downwash variation. Programming the horizontal tail incidence with wing tilt angle will have considerable influence on the variation of this derivative with tilt angle.

The rate of change of horizontal force with velocity is primarily dependent upon the wing forces at low speeds and again would primarily depend upon the tilt angle.

Typical variations of these two derivatives are shown in Figure 27. Experimental data are from Reference 25 and unpublished data from the Princeton dynamic model track. Figure 28 shows the variations used in the example in Chapter V.

The last two derivatives, the rate of change of vertical force with speed and the rate of change of horizontal force with attitude, tend generally to be made up of a number of small contributions, and are difficult to estimate. The latter derivative (X_w) is usually not too important and generally is quite nonlinear (Reference 25). Further experimental data are needed before detailed comments on these derivatives can be made.

We must also consider the existence and possible importance of the lag derivatives such as $M_{\dot{w}}$; however, the complex nature of the flow field around the tail, taken with the lack of experimental data, makes it difficult to estimate the possible importance of this quantity. Separated areas on the wing may contribute to this derivative (Reference 31), as well as the usual effect of the lag between a change in wing-propeller angle of attack and the time at which the downwash change is experienced at the tail.

If the tail is out of the downwash, as indicated by the experimental data for one configuration (Figure 26), this derivative is probably small at low speeds, due to the low dynamic pressure at the tail. It might be argued that at low speeds the effects would become more important, due to the longer time taken for the downwash to be propagated from the wing to the tail. This would, of course, depend upon whether the effects are moving downstream with the downwash velocity or the flight velocity. It should also be recalled that this derivative is only an approximation to a time delay effect, and thus, if the delay time is long, then this should be treated as a nonlinear effect.

This discussion indicates general trends of the stability derivatives of tilt-wing aircraft. It should be realized that the discussion is based on relatively limited experimental data.

APPENDIX IV

ROOT LOCUS TECHNIQUE FOR COMPLEX GAIN

Given an equation in root locus from (Reference 33),

$$\frac{K(\lambda - z_1)(\lambda - z_2)}{(\lambda - p_1)(\lambda - p_2)(\lambda - p_3)} = -1, \quad (\text{D-1})$$

where z_1 and z_2 represent the zeros and p_1 , p_2 , and p_3 represent the poles of the function, we wish to consider how the rules for sketching root loci are influenced by the fact that K is a complex number. We denote

$$(\lambda - z_1) = \alpha_1 e^{i\varphi_1}$$

$$(\lambda - p_1) = \beta_1 e^{i\psi_1}$$

$$K = k e^{i\eta},$$

and so we may write the equation D-1 in polar form as

$$\frac{k \alpha_1 \alpha_2}{\beta_1 \beta_2 \beta_3} \{ e^{i[\eta + \psi_1 + \psi_2 - \varphi_1 - \varphi_2 - \varphi_3]} \} = e^{i[2k+1]\pi}. \quad (\text{D-2})$$

The gain is therefore computed in the usual way by measuring the distances from the poles and zeros to the location of interest and computing the magnitude of the gain, k ;

$$k = \frac{\beta_1 \beta_2 \beta_3}{\alpha_1 \alpha_2}. \quad (\text{D-3})$$

The angle condition depends upon the argument of the gain; i. e.,

$$(\psi_1 + \psi_2) - (\varphi_1 + \varphi_2 + \varphi_3) = (2k + 1)\pi - \eta. \quad (\text{D-4})$$

Thus, we subtract the sum of the angles from the poles to the point in question from the sum of the angles from the zeros and equate the result to $(2k + 1)\pi - \eta$. The locus is drawn in a conventional fashion. The locus is no longer symmetrical about the real axis; however, since we are, in general, finding the roots of a polynomial with a complex coefficient,

complex roots are no longer necessarily conjugates. Given the locus for the complex gain K , the locus for the conjugate gain \bar{K} is the locus for the gain K reflected about the real axis.

The asymptotes for a locus with complex gain are found from the equation describing the angles of the straight line segments of the asymptotes.

We find the asymptotes by taking only the highest order term λ , to obtain

$$\frac{K}{\lambda^{p-z}} \cong e^{(2k+1)\pi} \quad \varphi = \frac{(2k+1)\pi - \eta}{z - p}, \quad k = 0, 1, 2, \dots, (z - p - 1) \quad (D-5)$$

where φ is the argument of λ , and η is the argument of K . z is the number of zeros, and p is the number of poles of the function on the left-hand side of equation D-1.

To obtain the point of intersection of the asymptotes on the real axis, we approximate equation D-1 by taking the two highest order terms in λ , noting that $p > z$, so that

$$\frac{k e^{i\eta}}{\lambda^{p-z} + (\sum \text{Real Parts of Zeros} - \sum \text{Real Parts of Poles}) \lambda^{p-z-1}} \cong -1.$$

These first two terms may be considered as approximating a system with multiple poles and zeros all at the same location, i. e.,

$$\frac{k e^{i\eta}}{(\lambda + c)^{p-z}} \cong -1,$$

and this location is the location of the intersection. It is thus unaffected by the complex gain and is located at what may be termed the center of gravity of the poles and zeros.

$$C = \frac{\sum \text{Re}(z) - \sum \text{Re}(p)}{p - z}.$$

In general, when the gain is complex such that the angle condition is other than 0 or π , there will be no locus on the real axis.

We may compute the departure angles as in the conventional locus, using the suitable angle condition for the gain.

In conclusion, therefore, we proceed as with a conventional root locus, just taking into account the special angle condition.

Various angle loci for systems with two poles and no zeros and three poles and no zeros are shown in Figures 29 and 30.

APPENDIX V

PHYSICAL INTERPRETATION OF THE ASYMPTOTIC SOLUTION

It is interesting to consider a physical interpretation of the asymptotic solution for mechanical systems.

An elegant interpretation of the asymptotic solution has been given in terms of wave transmission through a variable medium that is not particularly enlightening for our purposes (Reference 32).

Let us consider, then, the differential equation

$$\frac{d^2\theta}{dt^2} + b(t) \theta = 0 \quad (E-1)$$

as a force equation describing a unit mass suspended by a spring whose spring constant is varying with time.

The energy equation may be obtained by multiplying this equation by the velocity $d\theta/dt$, and integrating with respect to time to obtain

$$\frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2} b(t) \theta^2(t) = E_0 + \frac{1}{2} \int_0^t \theta^2(s) \frac{db}{ds} ds . \quad (E-2)$$

The first term on the left-hand side is the instantaneous value of the kinetic energy and the second term is the instantaneous potential energy. Their sum is equal to the initial value of the energy of the system plus an integral arising from the variable spring constant.

The integral in the last term indicates that the energy of the system is changed by the varying spring. That is, if we increase the spring constant by a small amount, Δb , suddenly at some displacement, θ_i , there will be an increment in energy added to the system equal to $\frac{1}{2} \theta_i^2 \Delta b$.

The energy equation before the change was

$$\frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2} b_0 \theta^2 = E_0 , \quad (E-3)$$

and after the change

$$\frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2} (b_o + \Delta b) \theta^2 = E_o + \frac{1}{2} \theta_i^2 \Delta b . \quad (E-4)$$

The amount of energy change depends upon the particular value of the displacement at the instant at which the spring constant changes.

The total energy of the system would remain constant if the spring constant change occurred at zero displacement. At any other value of the displacement, an energy increment would be added to the system, the maximum increment occurring when the spring constant change takes place at maximum displacement. Thus if the spring constant changes in a stepwise fashion, the variation in the energy of the system will depend upon the displacement of the system when the change occurs.

Now, let us examine the other extreme, when the coefficient $b(t)$ is a slowly varying function of time, or more precisely, $(1/b^{3/2}) db/dt$ is small. The energy at any instant of time is

$$E(t) = \frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2} b(t) \theta^2 . \quad (E-5)$$

The asymptotic solution to equation E-1, applicable when $(1/b^{3/2}) db/dt$ is small (Chapter III) is:

$$\theta = \frac{C_o}{b^{1/4}} \cos \left(\int^t b^{1/2} ds + \varphi \right) , \quad (E-6)$$

and therefore

$$\frac{d\theta}{dt} = -b^{1/4} \left[C_o \sin \left(\int^t b^{1/2} ds + \varphi \right) + \frac{1}{4} \frac{\dot{b}}{b^{3/2}} C_o \cos \left(\int^t b^{1/2} ds + \varphi \right) \right] . \quad (E-7)$$

The assumption that the coefficient, b , is slowly varying enables us to neglect the second term in equation E-7. Substitution of equations E-6 and E-7 into equation E-5 results in the following expression for the energy:

$$E(t) = b^{1/2} (t) C_o^2 , \quad (E-8)$$

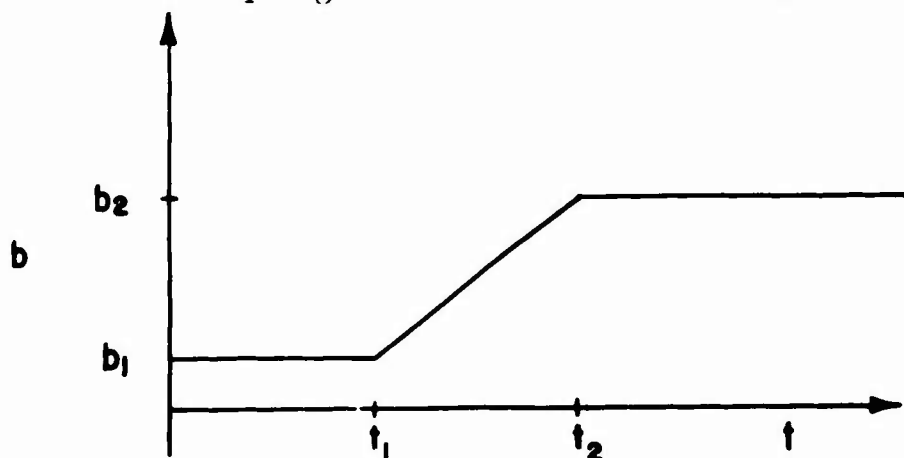
and therefore

$$\frac{E(t)}{b^{1/2}(t)} = \text{const.} = \frac{E(t)}{\omega(t)}. \quad (\text{E-9})$$

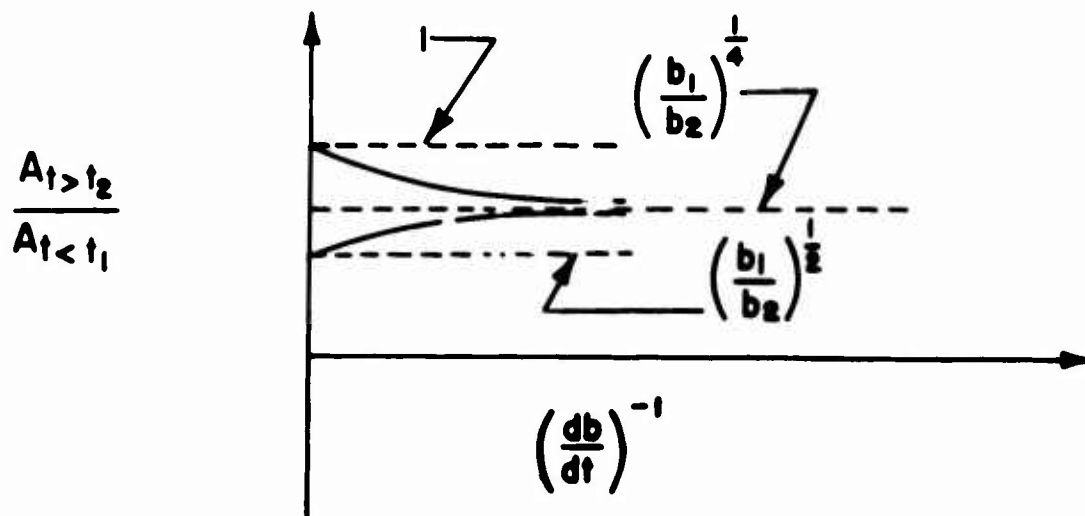
Thus, the energy divided by the frequency is constant in the slowly varying case. This is a result that was recognized many years ago in connection with the problem of a pendulum with slowly increasing or decreasing length (Reference 34).

Note that the latter case, with slowly varying coefficients, is reversible, the amplitude of the response is only a function of the magnitude of the spring constant, where in the former case of the very rapidly changing spring constant, the amplitude variations depend upon the instant at which the spring constant changes.

These points are illustrated by the following example. Consider a system which has a spring constant variation with time as follows:



The ratio of the amplitude at $t > t_2$ to that at $t < t_1$ can be displayed in the following graph as:



The shaded area indicates possible displacement amplitudes for $t > t_2$ compared to the amplitude $t < t_1$, indicating the dependence of the final amplitude ($t > t_2$) upon the magnitude of the displacement when the change occurs. As $b \rightarrow \infty$, the limiting cases are:

1. The spring constant is changed when $\theta = \theta_{\max}$. So θ_{\max} remains constant and there is no change in the amplitude. Maximum energy input.
2. The spring constant is changed when $\theta = 0$, and so the energy remains constant and the displacement amplitude changes proportional to \sqrt{b} .

This simple example illustrates that when we leave the range of parameters over which the asymptotic solution applies, we can no longer expect to describe the response by a single amplitude function times oscillatory terms of unit magnitude. This may, in many cases, be due to the fact that the system is not oscillatory.

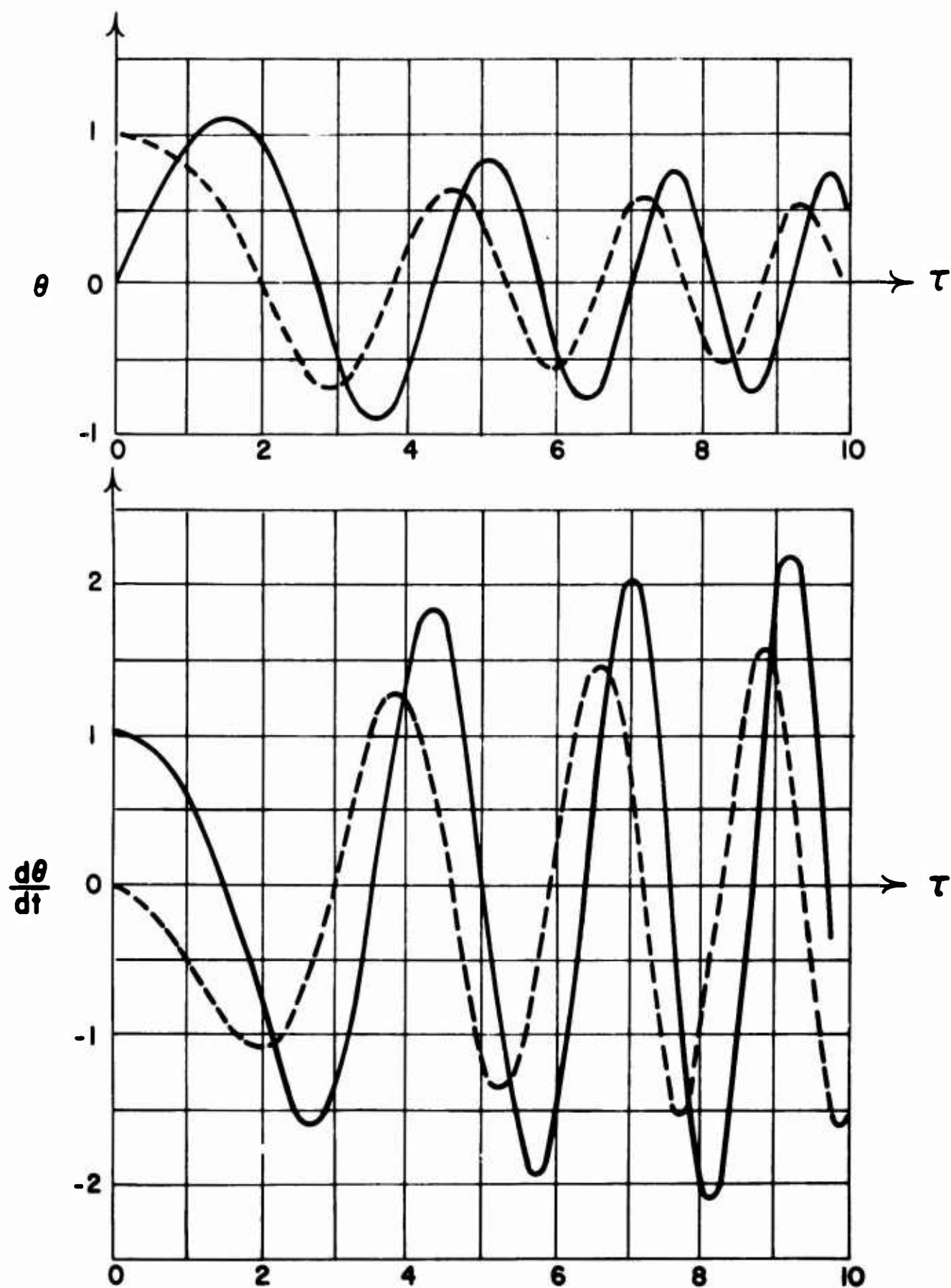


FIGURE 1 SOLUTIONS TO AIRY'S EQUATION

$$\left(\frac{d^2\theta}{d\tau^2} + \tau\theta = 0\right)$$

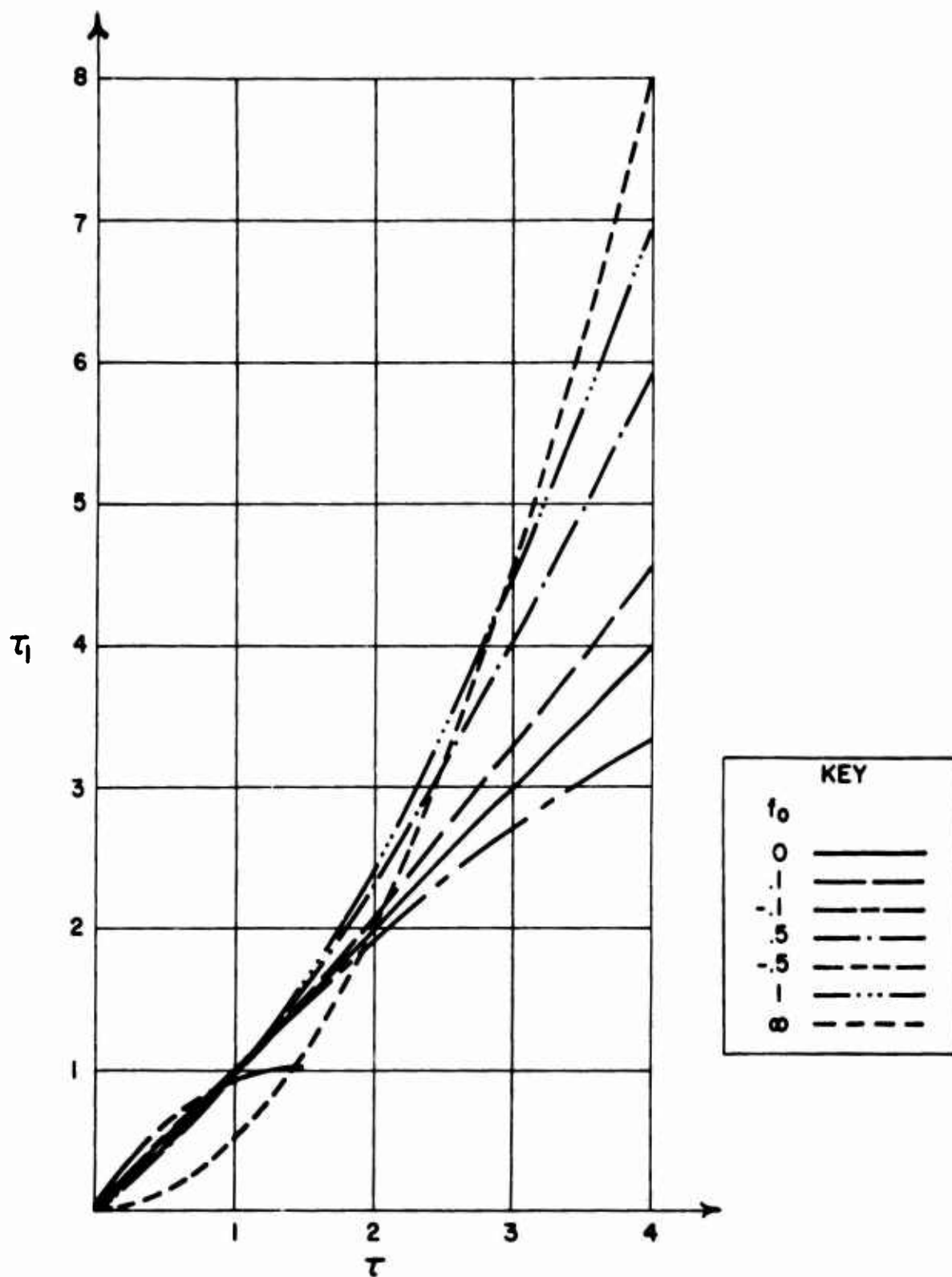


FIGURE 2 THE TRANSFORMED INDEPENDENT VARIABLE, τ_1 , VERSUS NON-DIMENSIONAL REAL TIME, τ , FOR CONSTANT ACCELERATION (EQUATION V-28), AS A FUNCTION OF f_0

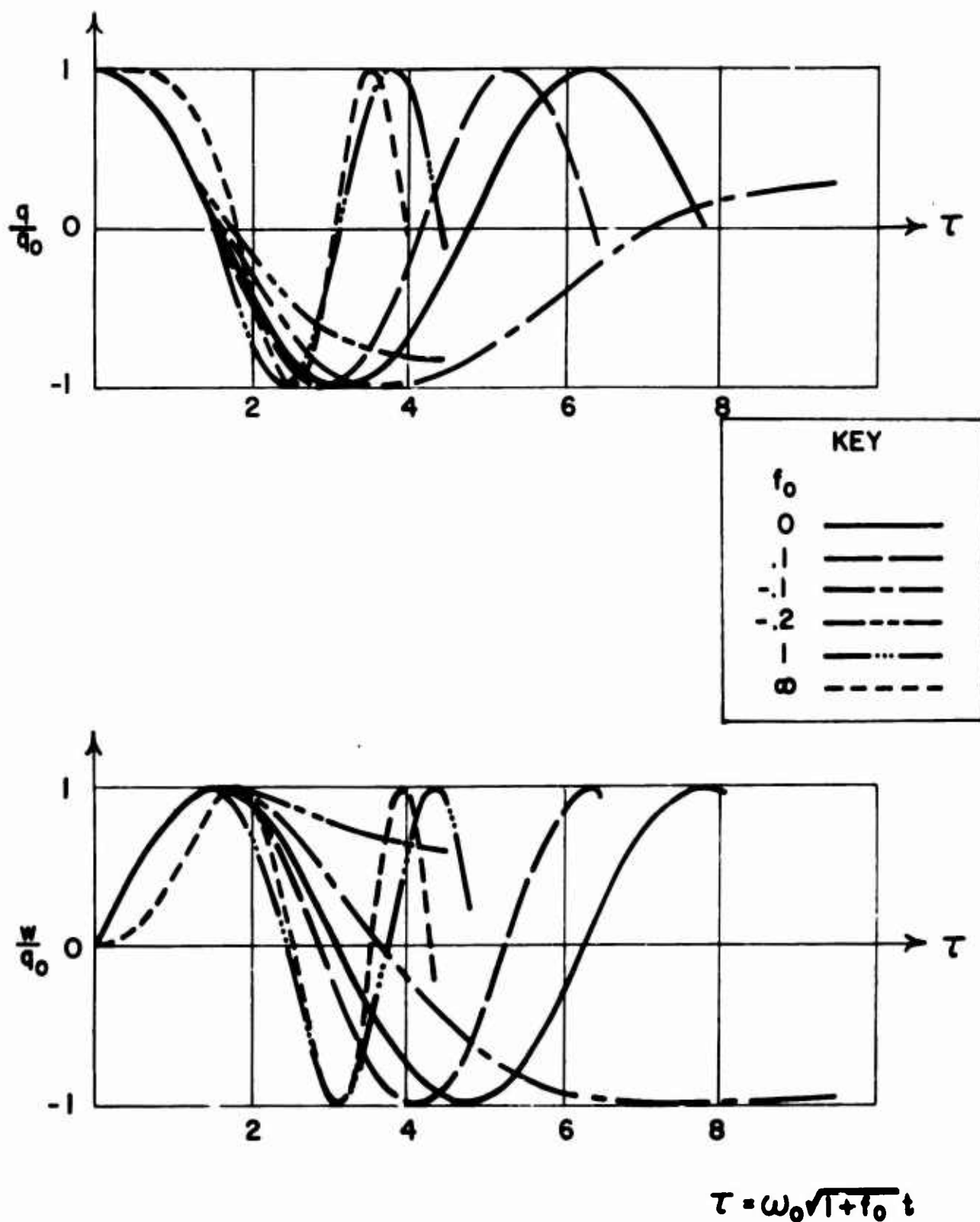
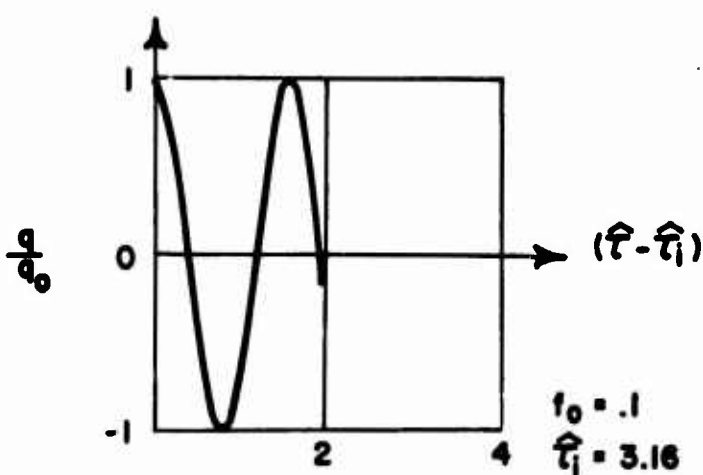
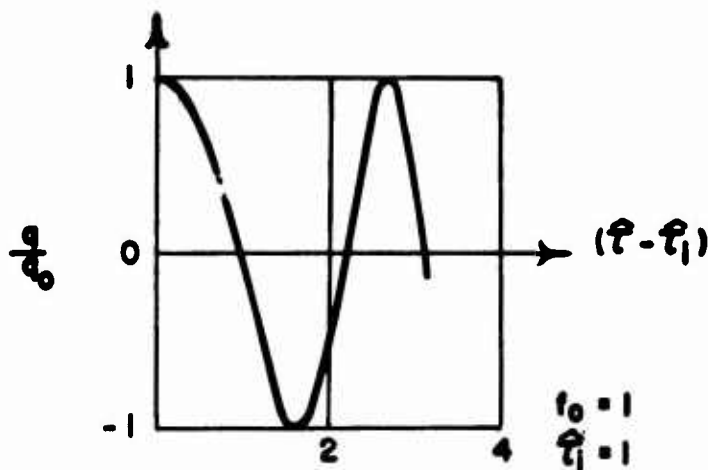
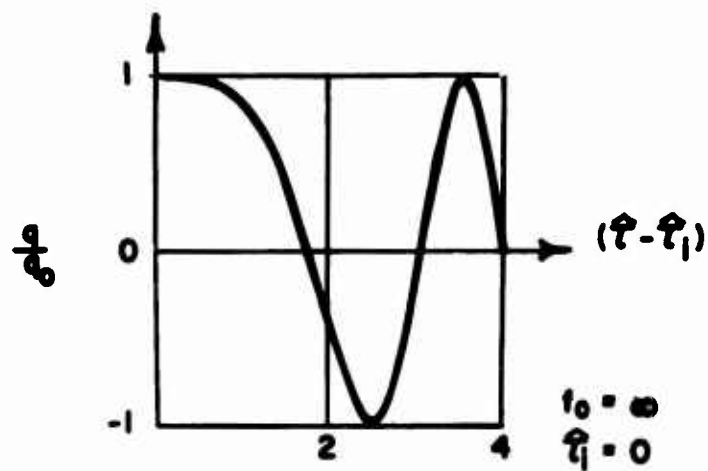


FIGURE 3 TRANSIENT RESPONSE FOR VARIOUS VALUES OF
THE ACCELERATION PARAMETER, f_0
CONSTANT ACCELERATION
ATTITUDE STABILITY PROPORTIONAL TO FLIGHT SPEED
NO AIRFRAME DAMPING



NOTE THE DEFINITION
OF NON-DIMENSIONAL
TIME:
 $\tau = \sqrt{\frac{a_0}{f_0}} t$

FIGURE 4 CHANGE IN TRANSIENT RESPONSE WITH TIME
DURING ACCERATION FROM HOVER
CONSTANT ACCELERATION

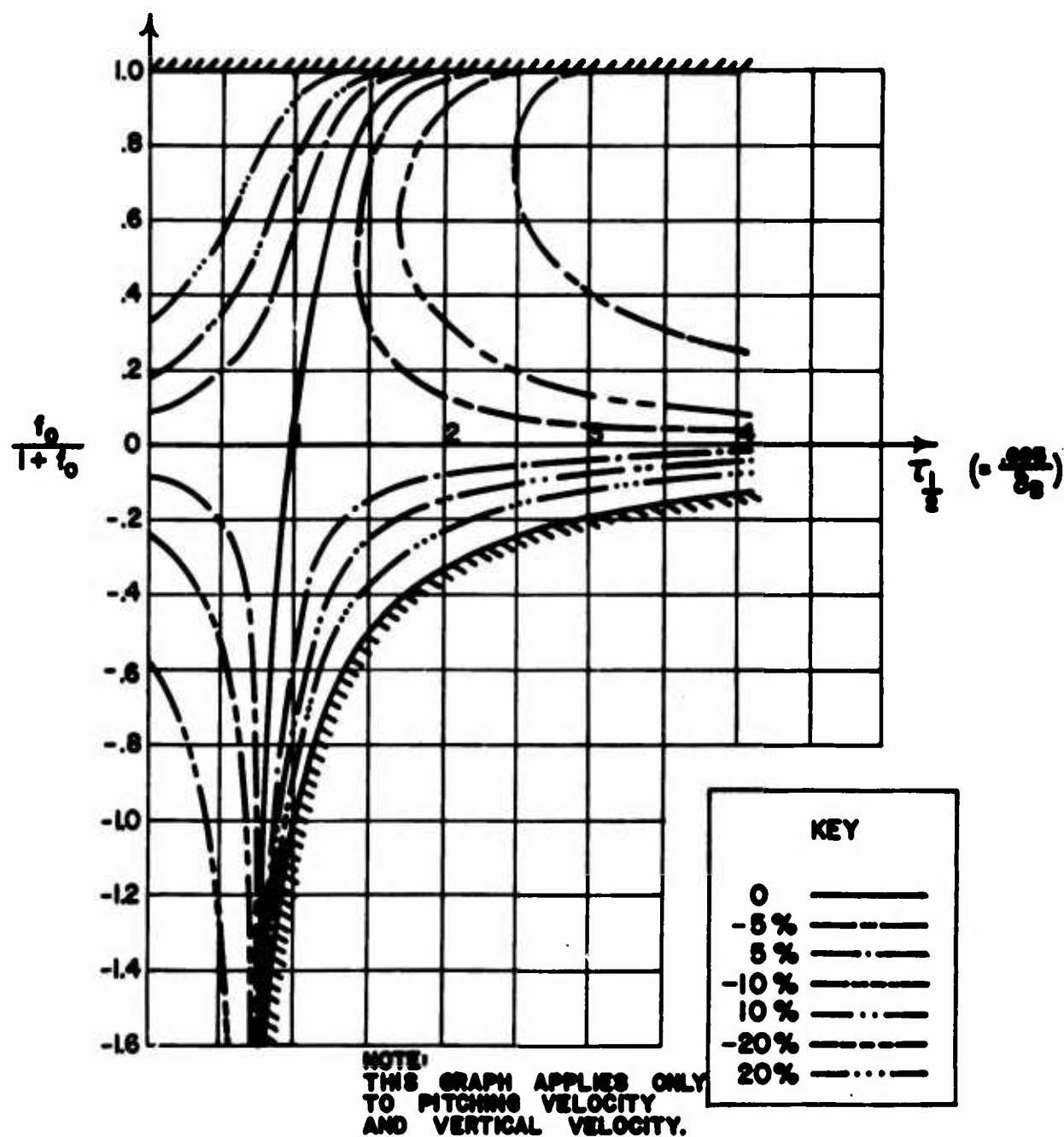


FIGURE 5 PERCENTAGE CHANGE IN THE TIME TO ONE - HALF AMPLITUDE AS A FUNCTION OF THE STEADY FLIGHT DAMPING RATIO, δ_0 , AND THE ACCELERATION PARAMETER, t_0
CONSTANT ACCELERATION

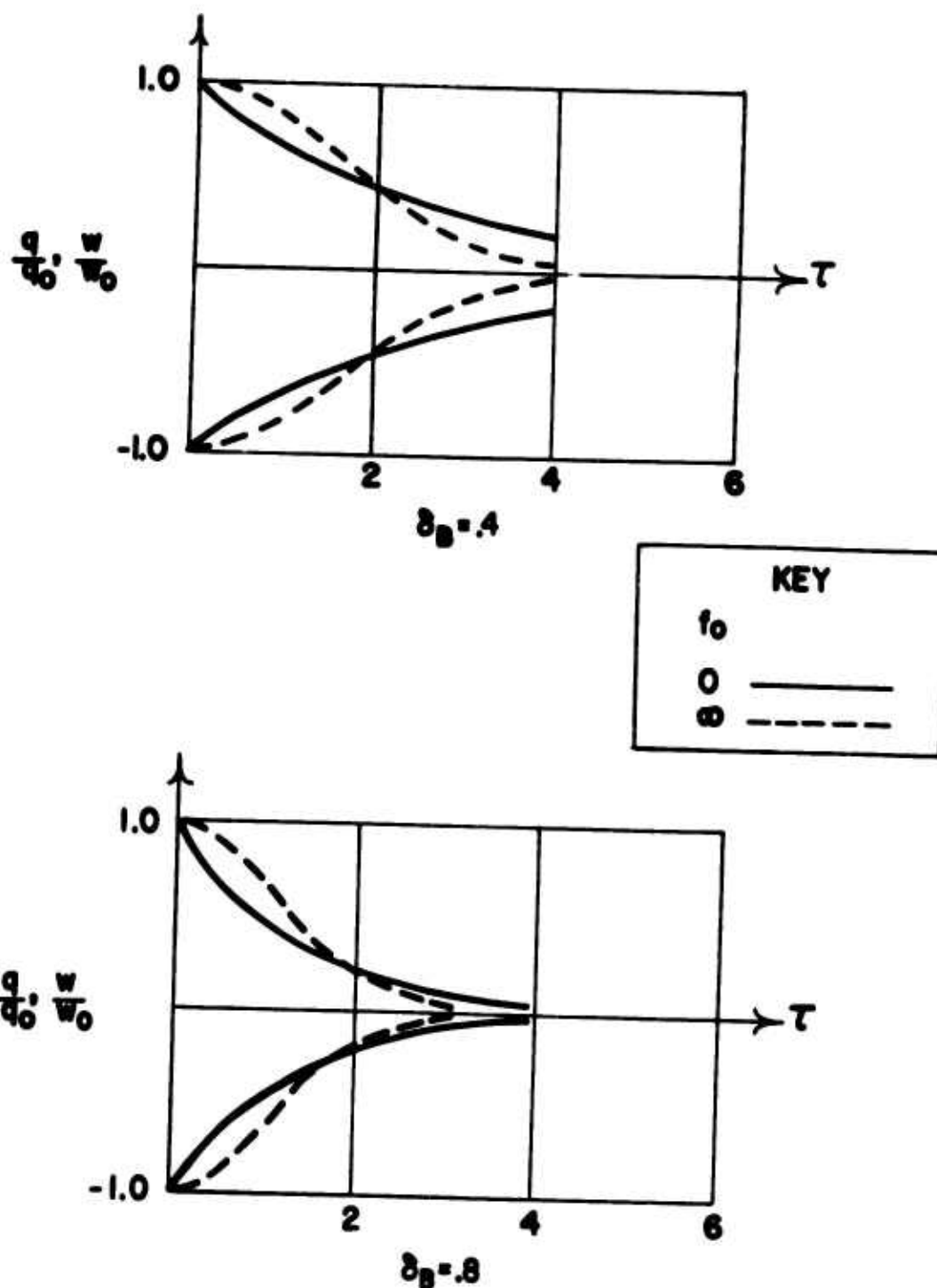


FIGURE 6 COMPARISON OF RESPONSE ENVELOPES FOR TWO STEADY FLIGHT DAMPING RATIOS, WITH $f_0 = 0$ AND $f_0 = \infty$ CONSTANT ACCELERATION

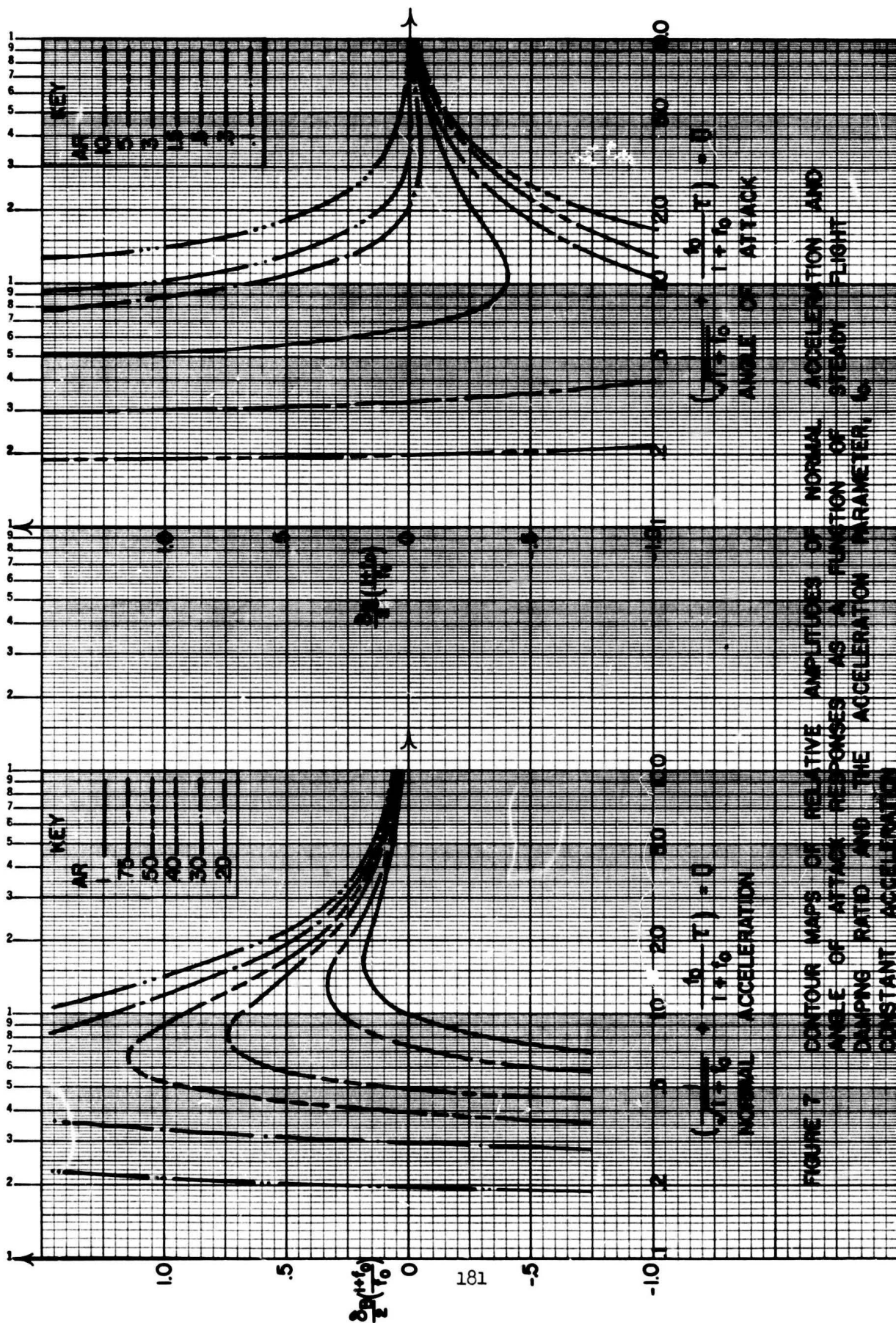


FIGURE 7 CONTOUR MAPS OF RELATIVE AMPLITUDES OF NORMAL ACCELERATION AND ANGLE OF ATTACK RESPONSES AS A FUNCTION OF STEADY FLIGHT DAMPING RATIO AND THE ACCELERATION PARAMETER, b

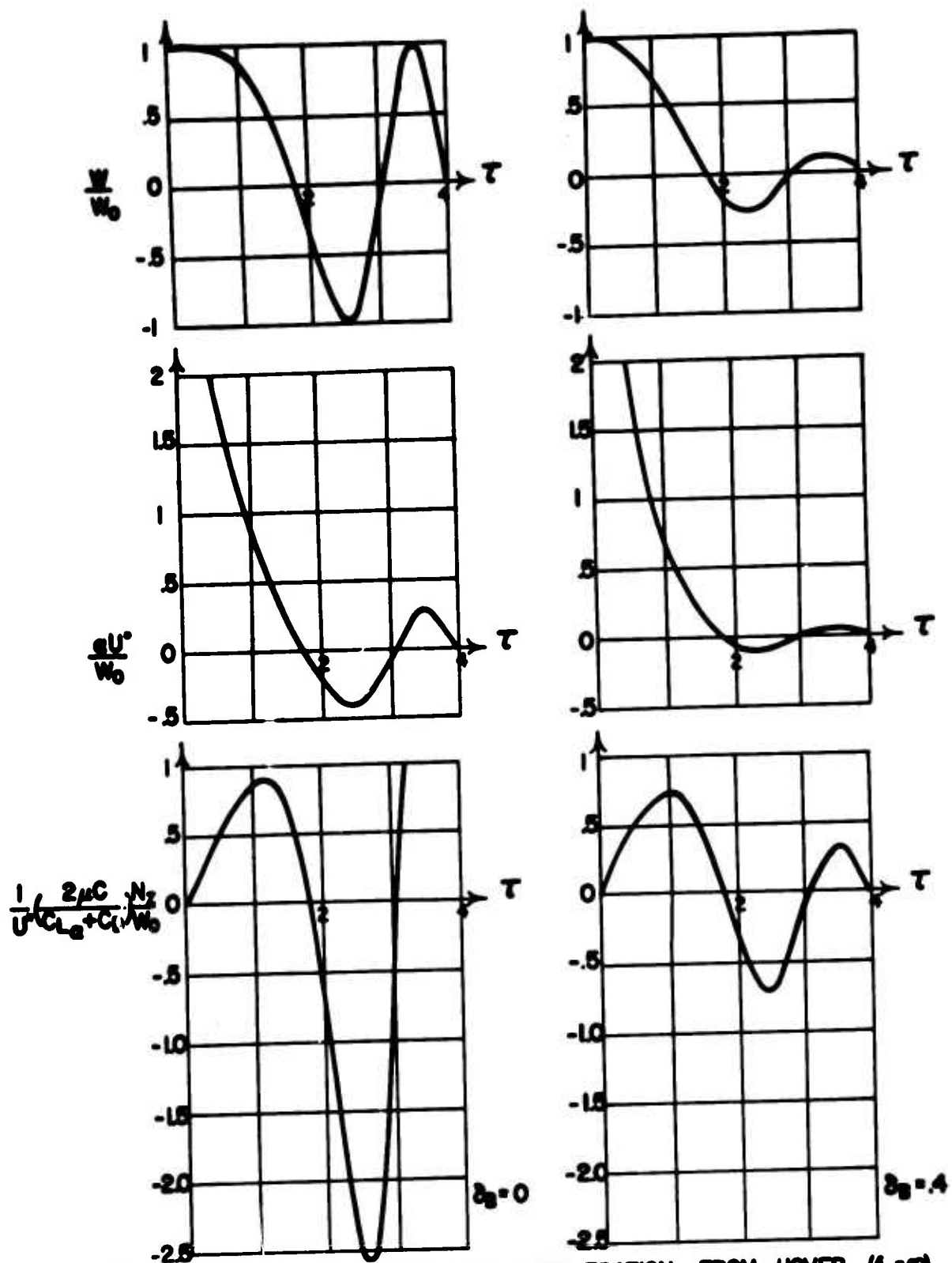


FIGURE 8 TRANSIENT RESPONSE FOR ACCELERATION FROM HOVER ($\dot{f}_0 = \infty$)
 WITH STEADY FLIGHT DAMPING RATIOS OF 0 AND 4
 ATTITUDE STABILITY PROPORTIONAL TO FLIGHT SPEED
 CONSTANT ACCELERATION

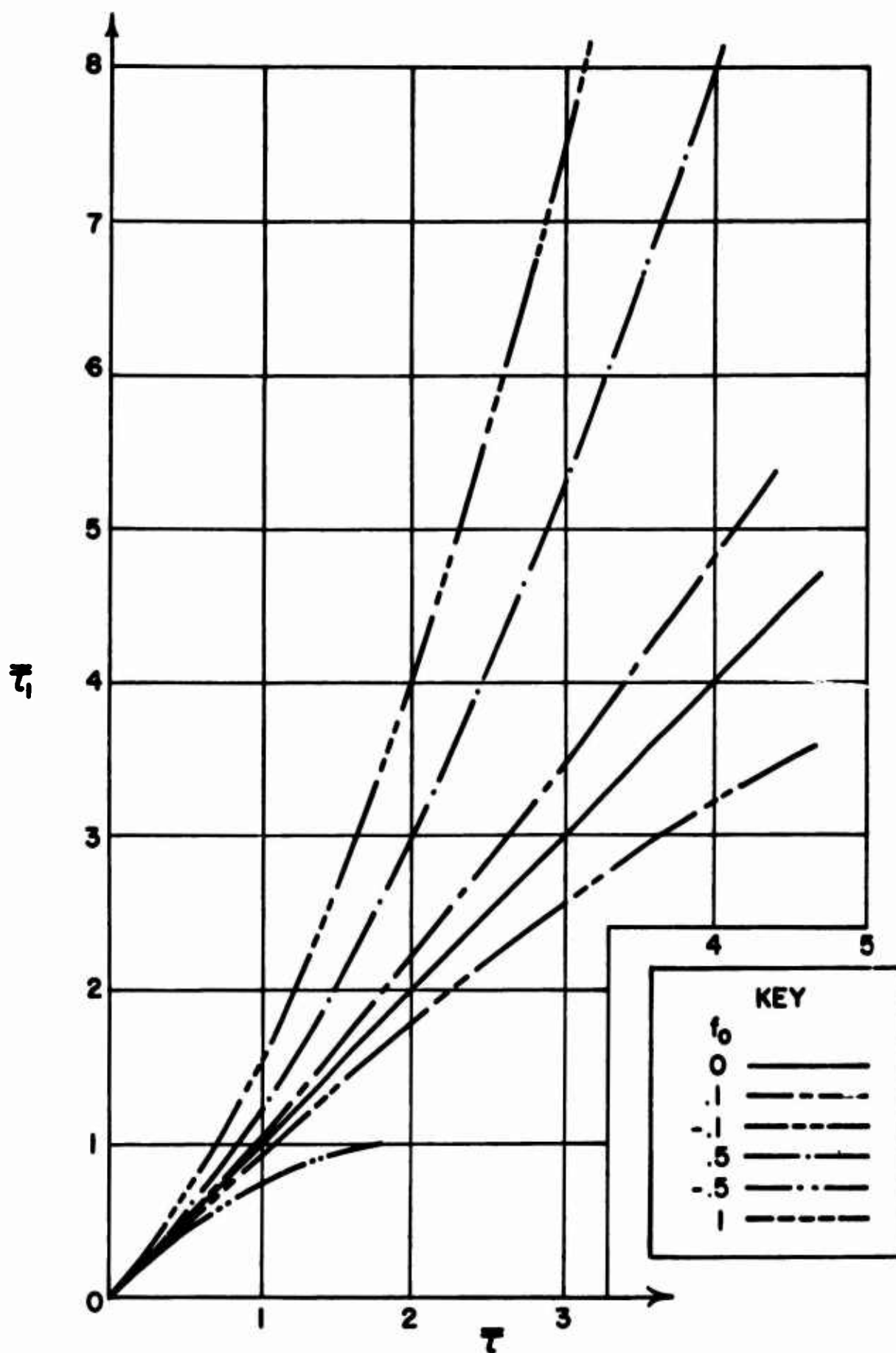


FIGURE 9 THE TRANSFORMED INDEPENDENT VARIABLE, $\bar{\tau}_1$, VERSUS NON-DIMENSIONAL REAL TIME, $\bar{\tau}$, FOR CONSTANT ACCELERATION, AS A FUNCTION OF \bar{t}_0

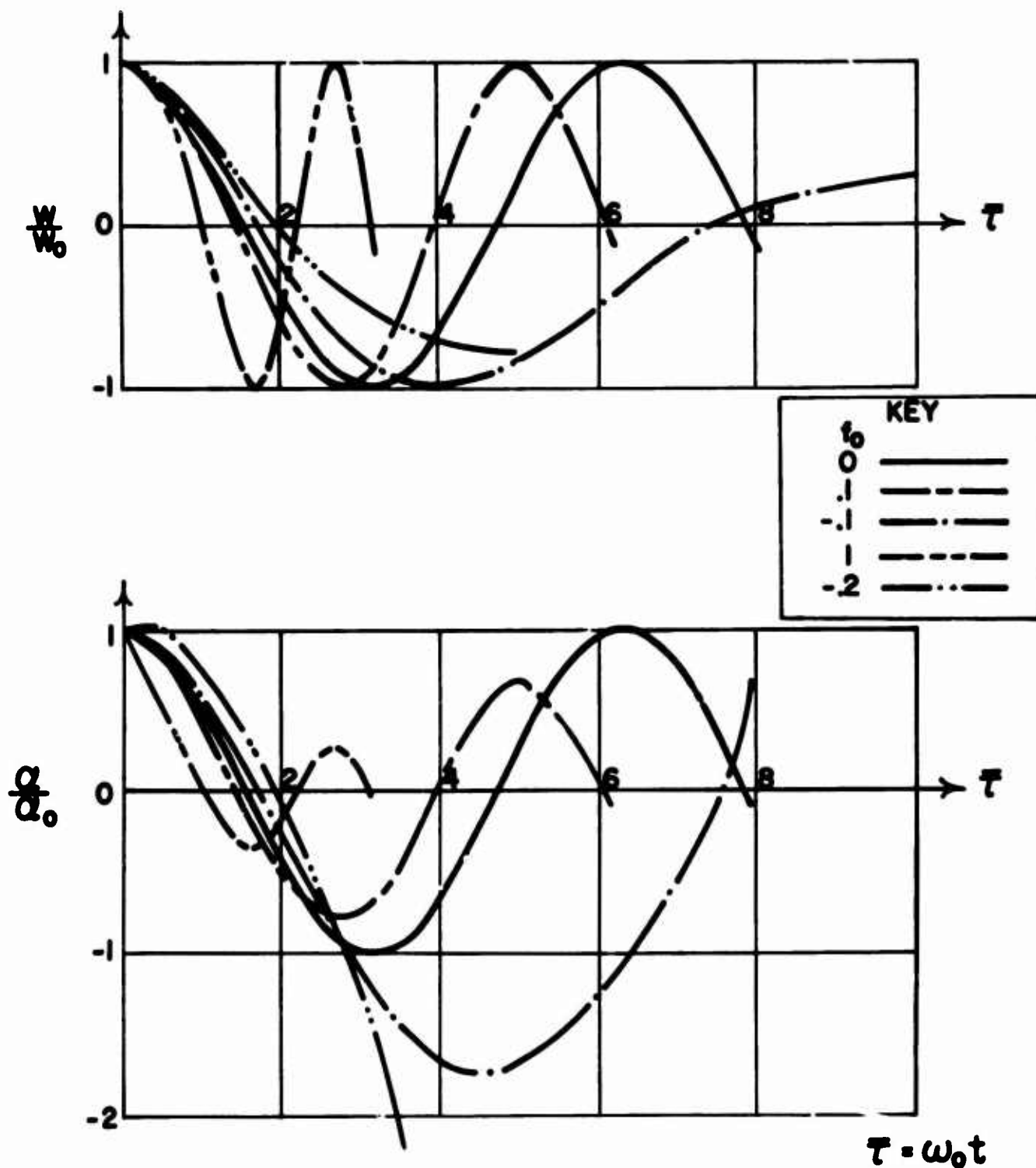


FIGURE 10 TRANSIENT RESPONSE FOR VARIOUS VALUES OF THE ACCELERATION PARAMETER, f_0

CONSTANT ACCELERATION

ATTITUDE STABILITY PROPORTIONAL TO FLIGHT SPEED

NO AIRFRAME DAMPING

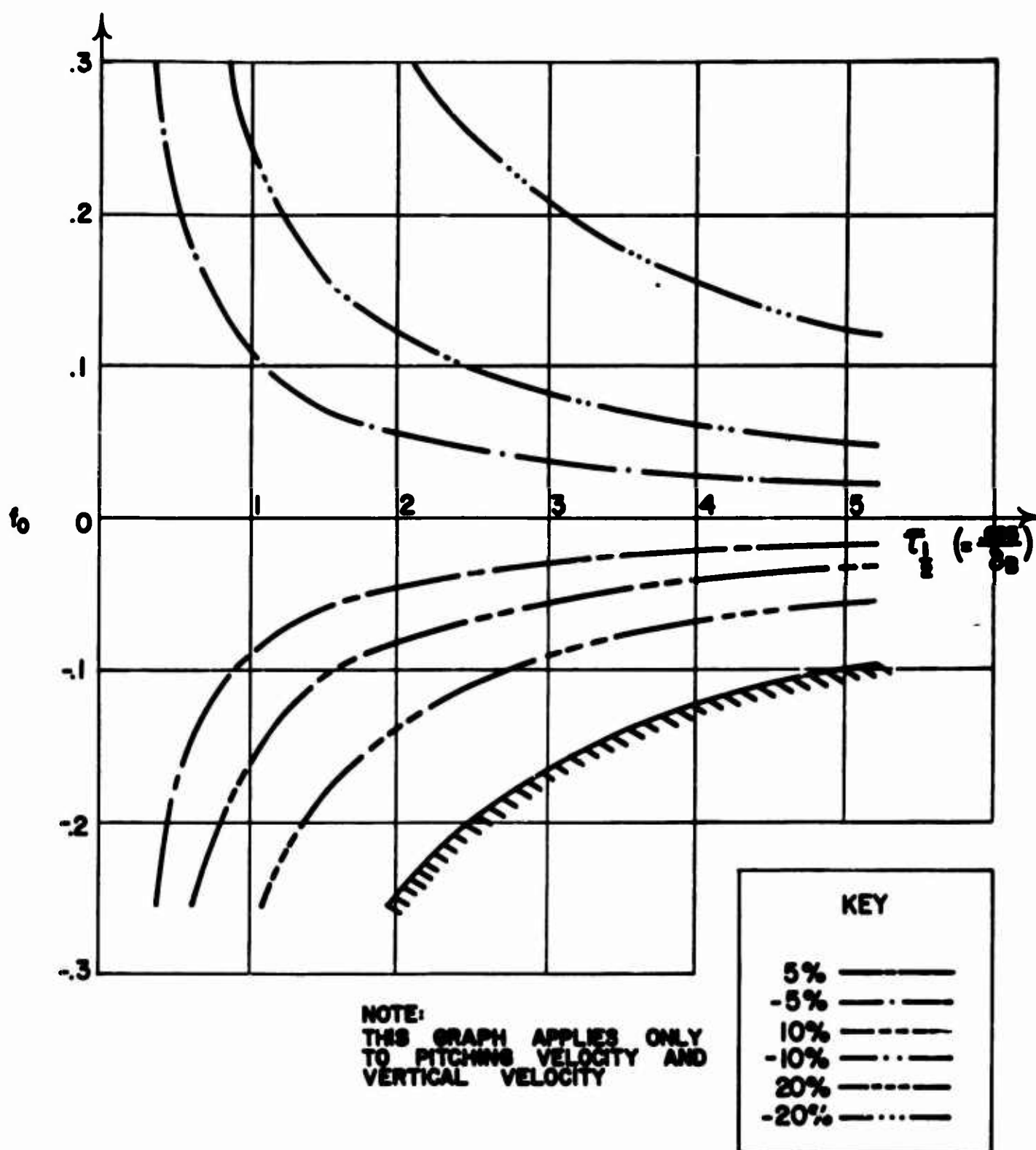


FIGURE II PERCENTAGE CHANGE IN THE TIME TO ONE-HALF AMPLITUDE AS A FUNCTION OF THE STEADY FLIGHT DAMPING RATIO, δ_g , AND THE ACCELERATION PARAMETER, f_0
CONSTANT ACCELERATION

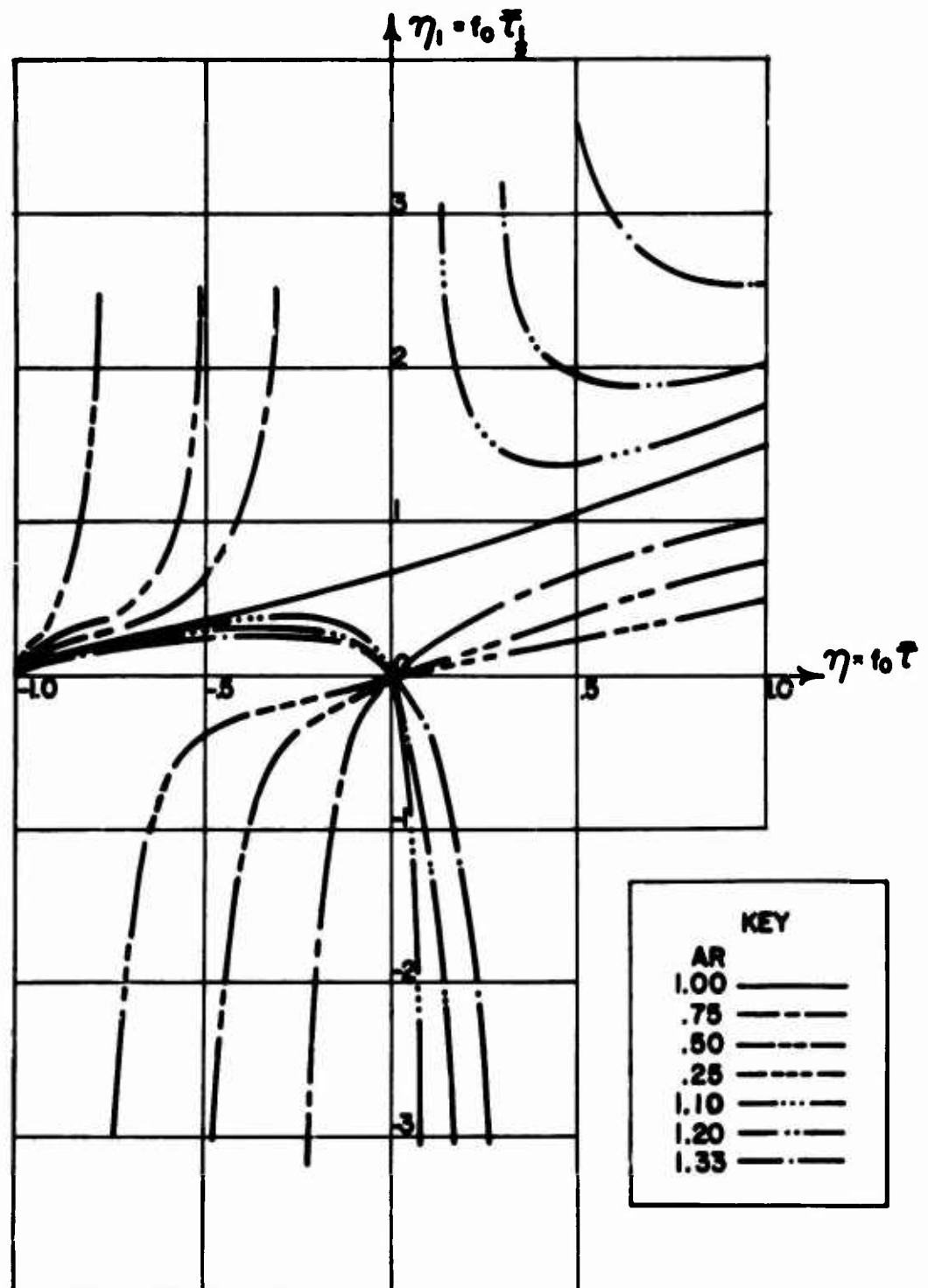


FIGURE 12a CONTOUR MAP OF RELATIVE AMPLITUDE OF NORMAL ACCELERATION RESPONSE AS A FUNCTION OF THE ACCELERATION PARAMETER, η_0 , AND THE TIME TO ONE-HALF AMPLITUDE IN STEADY FLIGHT, τ_1 . CONSTANT ACCELERATION

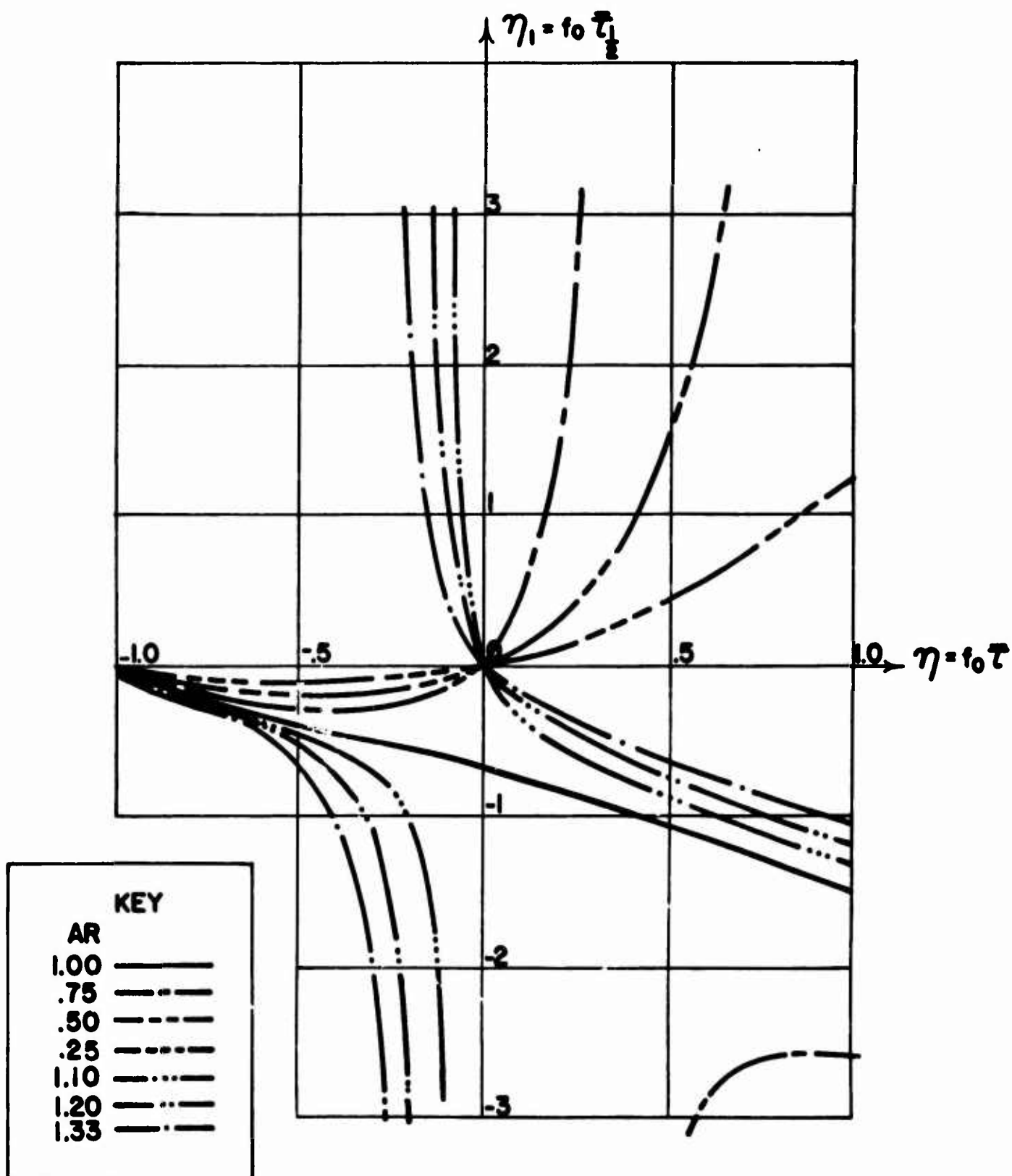


FIGURE 12b CONTOUR MAP OF RELATIVE AMPLITUDE OF ANGLE OF ATTACK RESPONSE AS A FUNCTION OF THE ACCELERATION PARAMETER, f_0 , AND THE TIME TO ONE-HALF AMPLITUDE IN STEADY FLIGHT, $\tau_{\frac{1}{2}}$
CONSTANT ACCELERATION

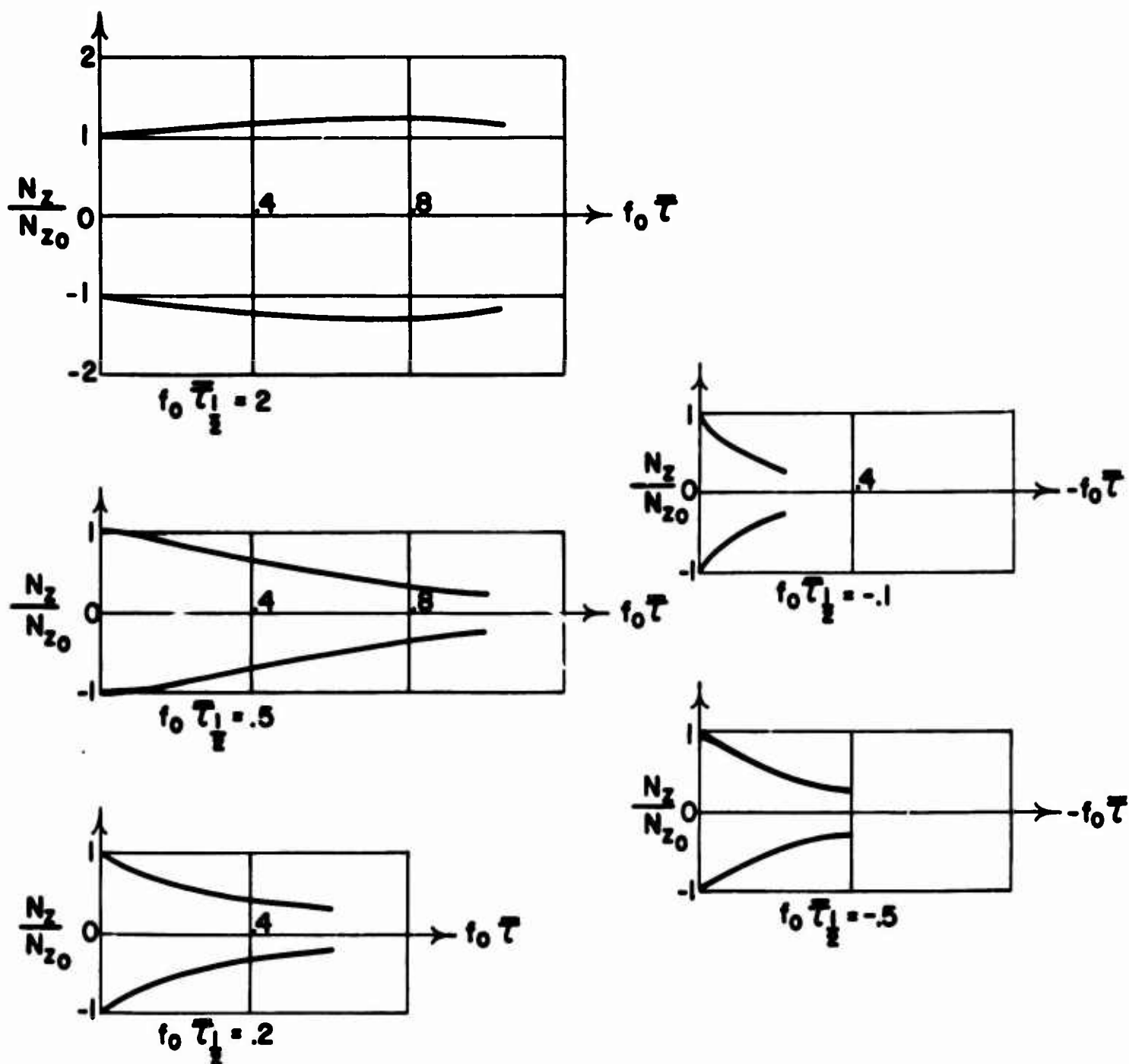


FIGURE 12c

AMPLITUDE OF NORMAL ACCELERATION RESPONSE
VERSUS TIME FOR VARIOUS VALUES OF THE
ACCELERATION PARAMETER, f_0 , AND THE TIME
TO ONE-HALF AMPLITUDE IN STEADY FLIGHT, $T_{1/2}$

CONSTANT ACCELERATION

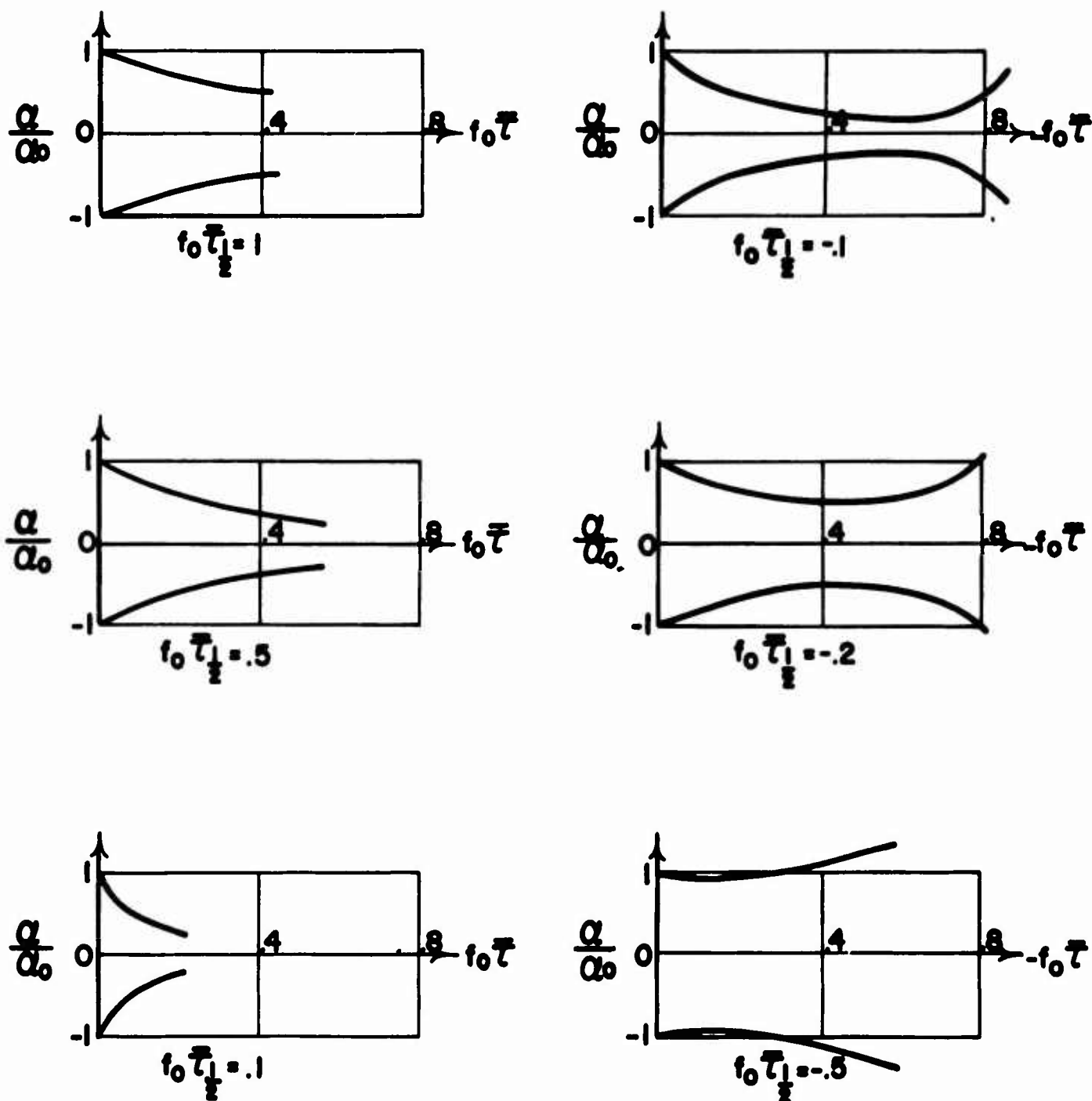


FIGURE 12d AMPLITUDE OF ANGLE OF ATTACK RESPONSE
VERSUS TIME FOR VARIOUS VALUES OF THE
ACCELERATION PARAMETER, f_0 , AND THE TIME
TO ONE-HALF AMPLITUDE IN STEADY FLIGHT, $T_{1/2}$
CONSTANT ACCELERATION

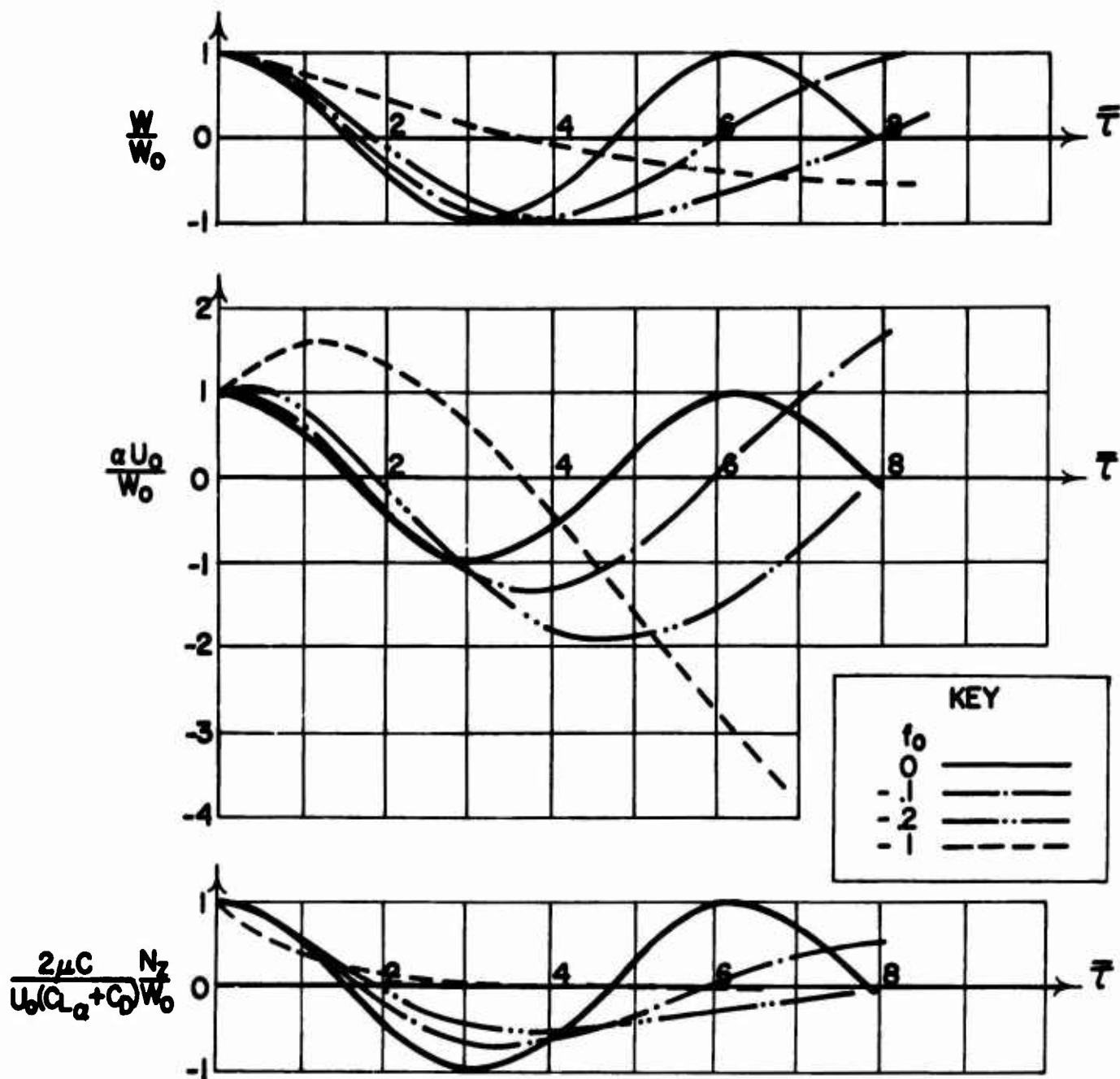


FIGURE 13a TRANSIENT RESPONSE FOR VARIOUS VALUES OF THE ACCELERATION PARAMETER, f_0

DECELERATION DUE TO DRAG

ATTITUDE STABILITY PROPORTIONAL TO FLIGHT SPEED

NO AIRFRAME DAMPING

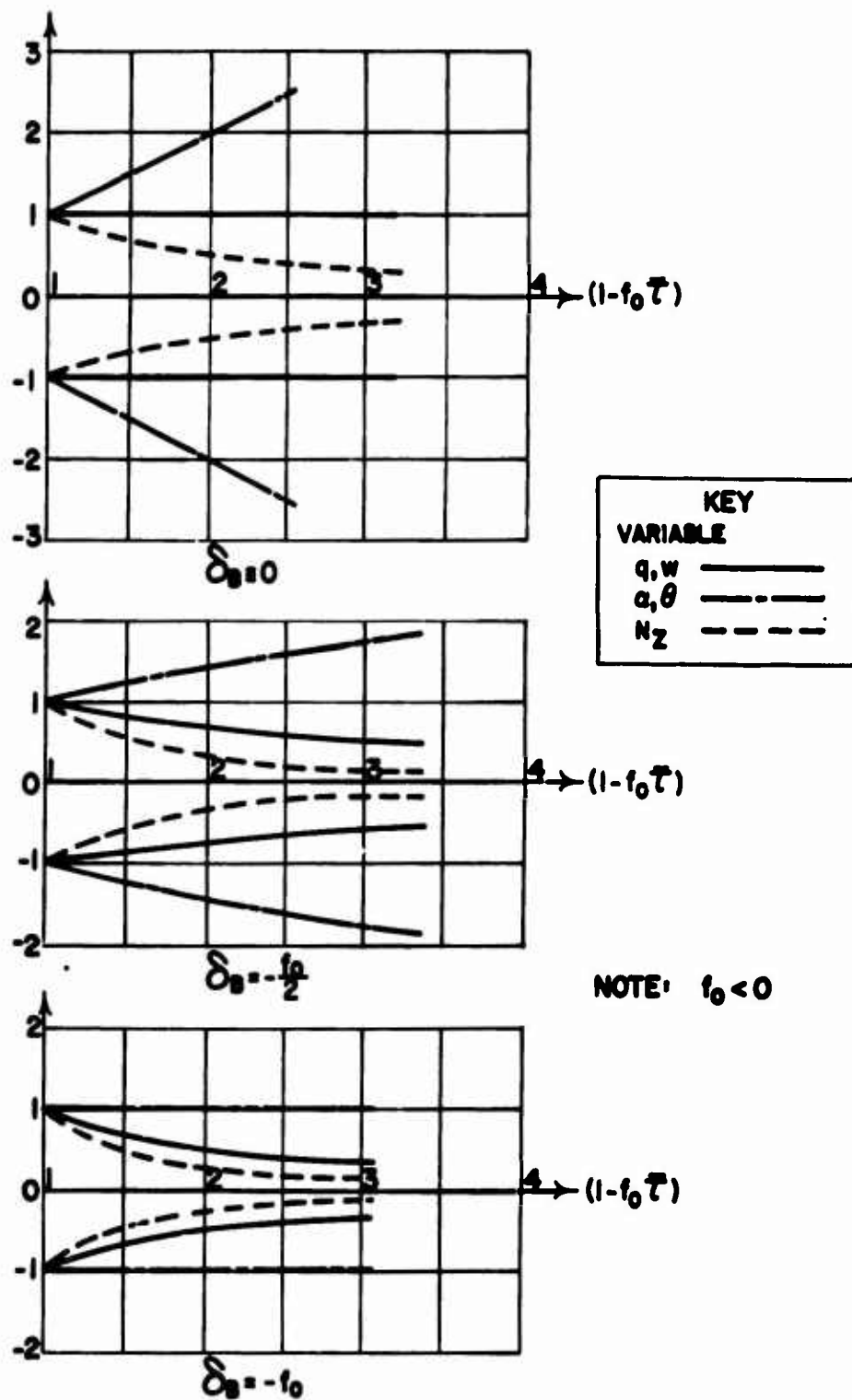


FIGURE 13b DECELERATION DUE TO DRAG
COMPARISON OF VARIABLE ENVELOPES
LOW AIRFRAME DAMPING

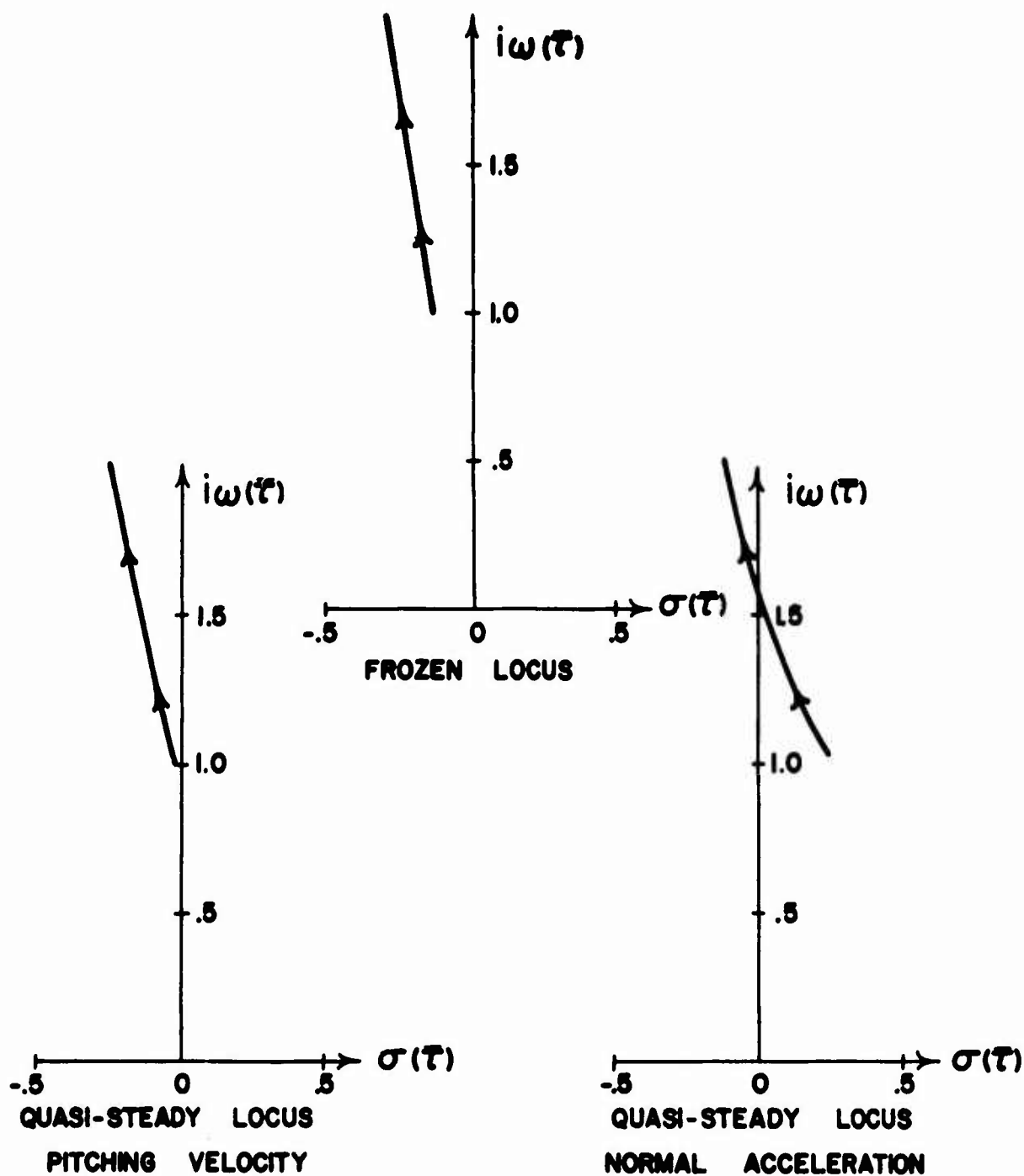


FIGURE 14a COMPLEX PLANE INTERPRETATION OF TIME VARYING EFFECTS

STABILITY DERIVATIVES PROPORTIONAL TO FLIGHT SPEED

CONSTANT ACCELERATION, $f_0 = .25$

$\delta_B = .15$

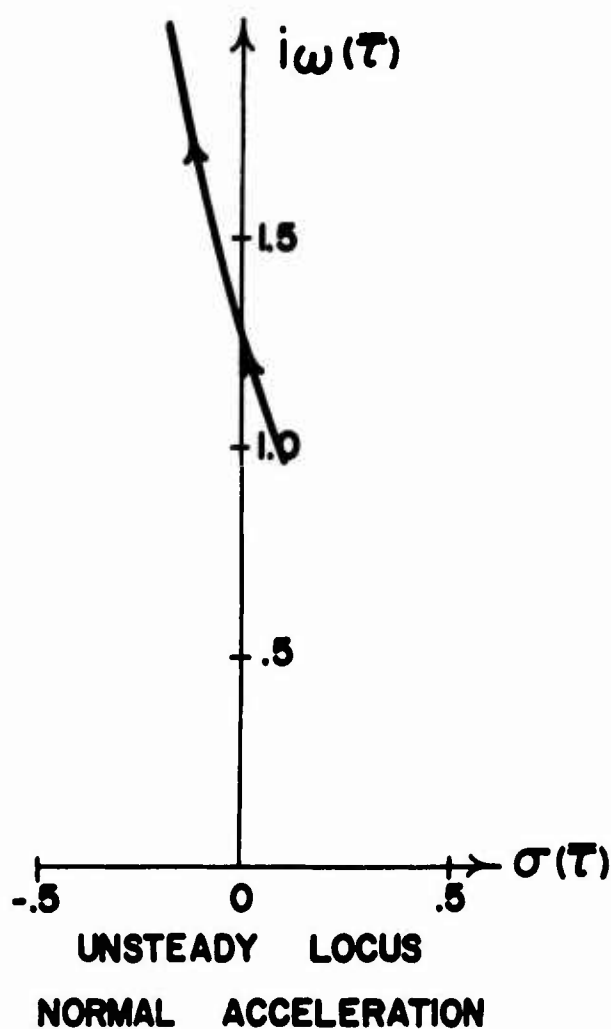
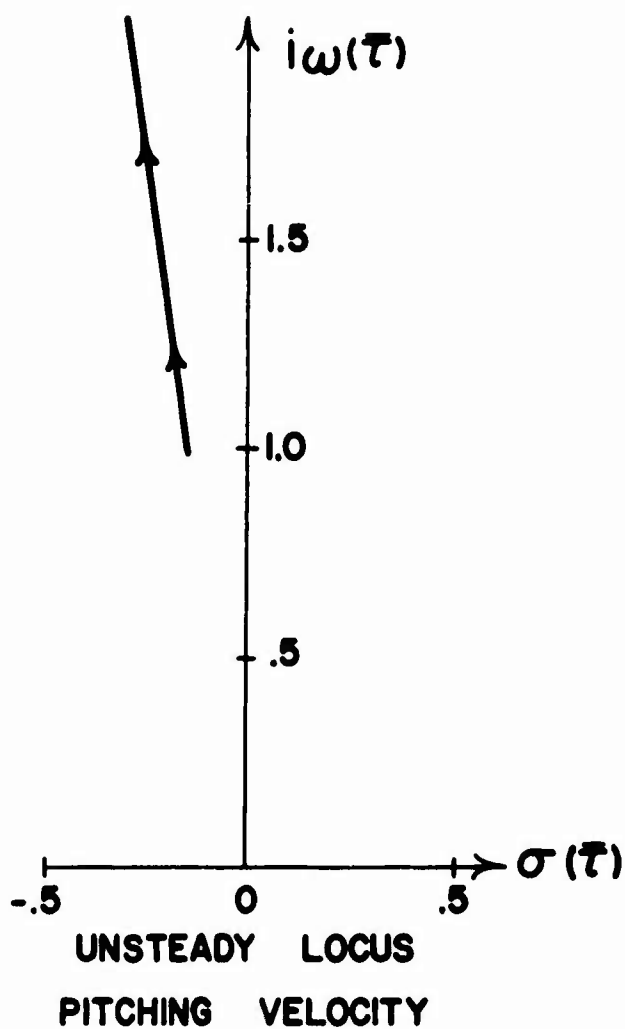


FIGURE 14b COMPLEX PLANE INTERPRETATION OF TIME VARYING EFFECTS

STABILITY DERIVATIVES PROPORTIONAL TO FLIGHT SPEED

CONSTANT ACCELERATION, $f_0 = .25$

$\delta_B = .15$

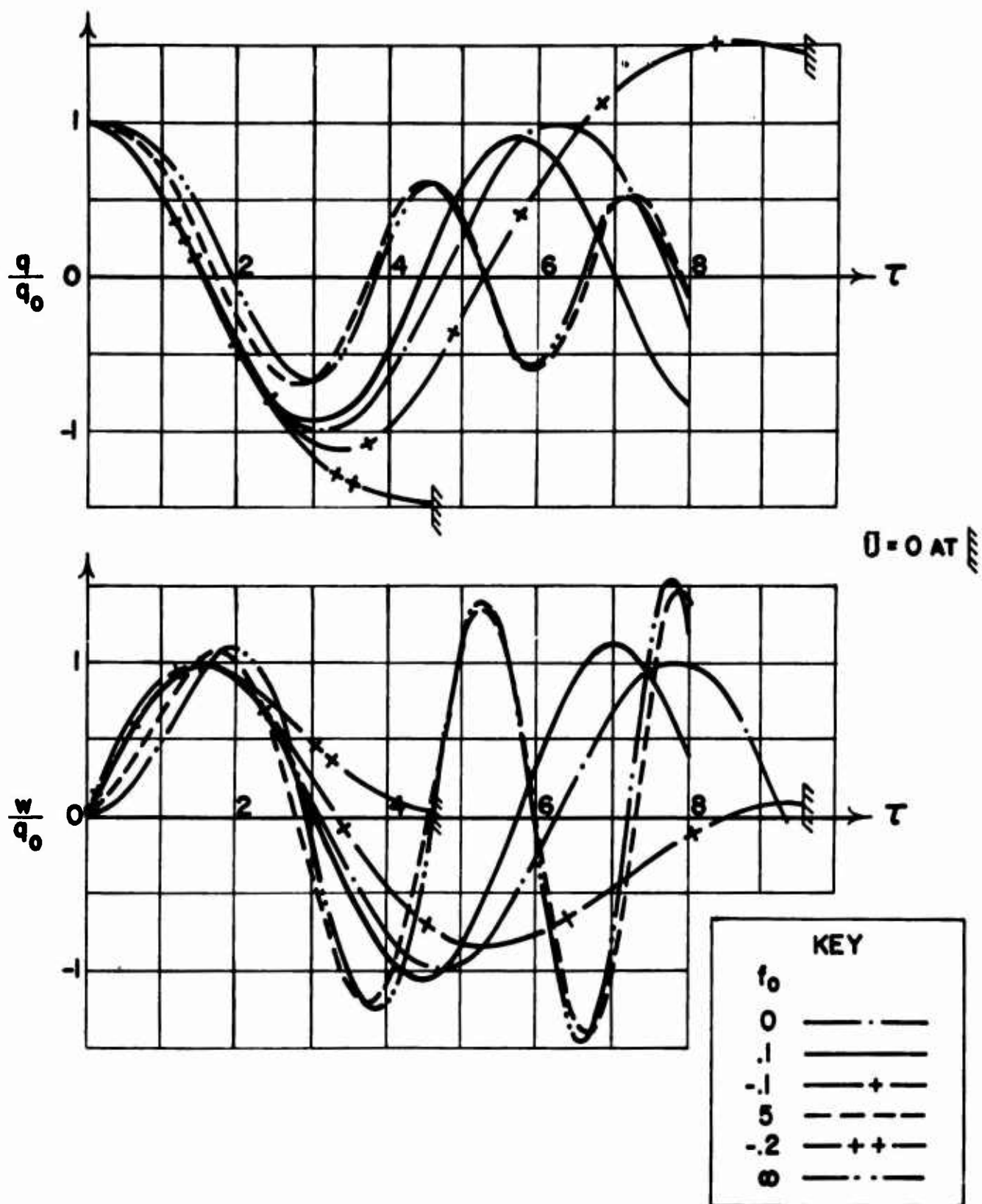


FIGURE 15 **TRANSIENT RESPONSE FOR VARIOUS VALUES**
OF THE ACCELERATION PARAMETER, f_0
CONSTANT ACCELERATION
ATTITUDE STABILITY CONSTANT
NO AIRFRAME DAMPING

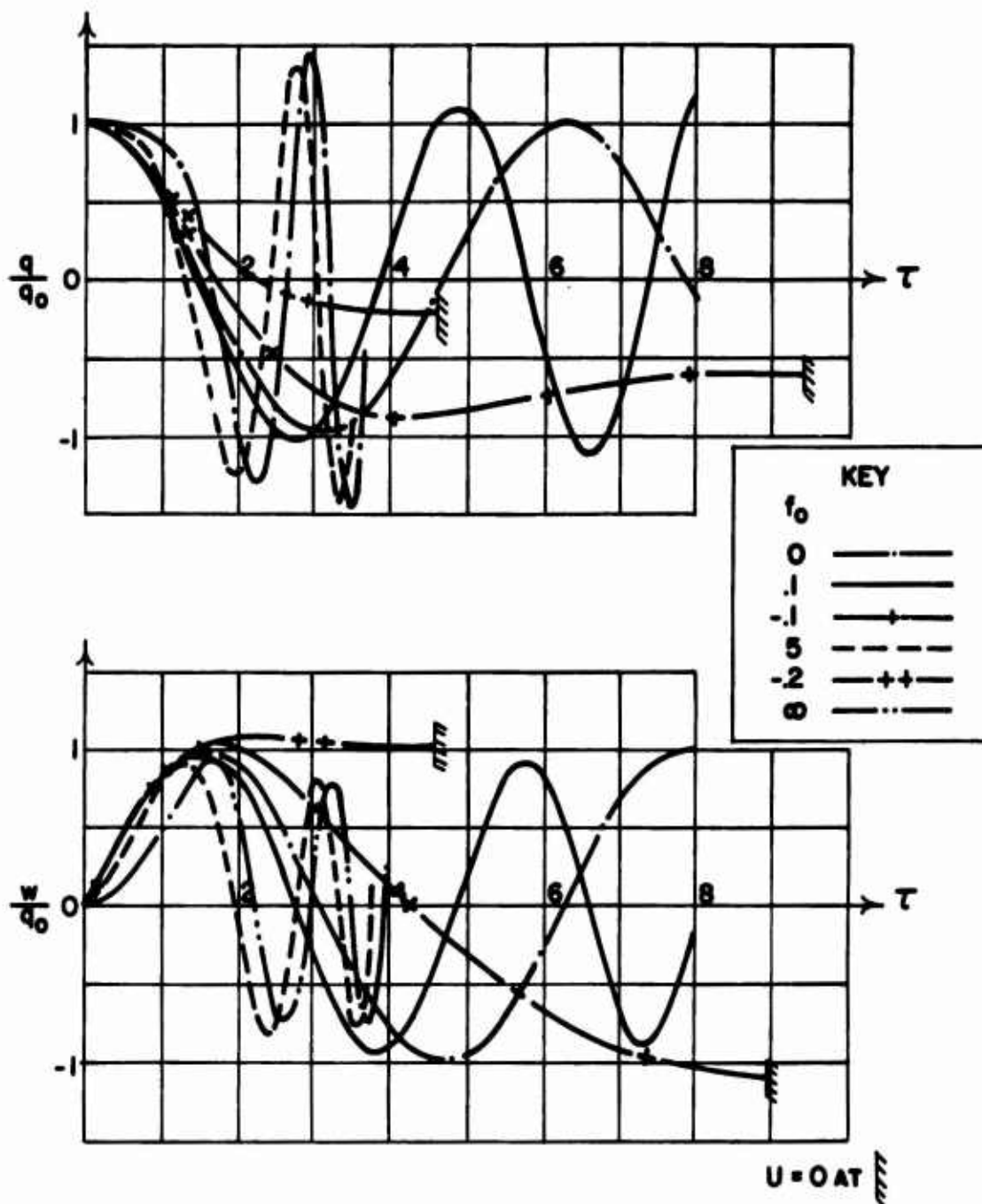


FIGURE 16 TRANSIENT RESPONSE FOR VARIOUS VALUES OF THE ACCELERATION PARAMETER, f_0

CONSTANT ACCELERATION

ATTITUDE STABILITY PROPORTIONAL TO THE SQUARE OF THE FLIGHT SPEED

NO AIRFRAME DAMPING

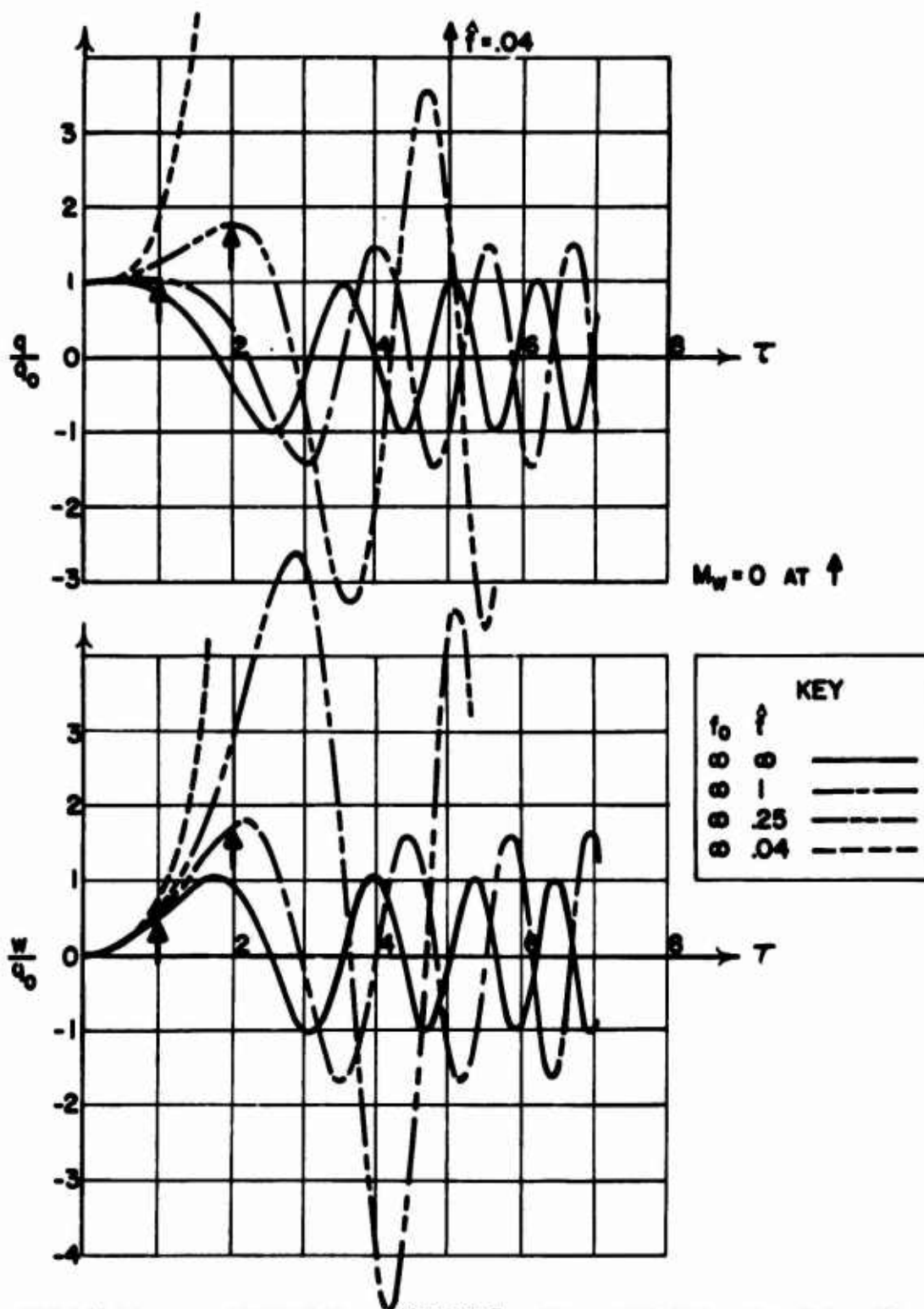


FIGURE 17b TRANSIENT RESPONSE FOR VARIOUS VALUES OF THE ACCELERATION PARAMETER \hat{f} , WHEN $f_0 = \infty$

CONSTANT ACCELERATION

ATTITUDE STABILITY PROPORTIONAL TO SPEED AND NON-ZERO IN HOVER

NO AIRFRAME DAMPING

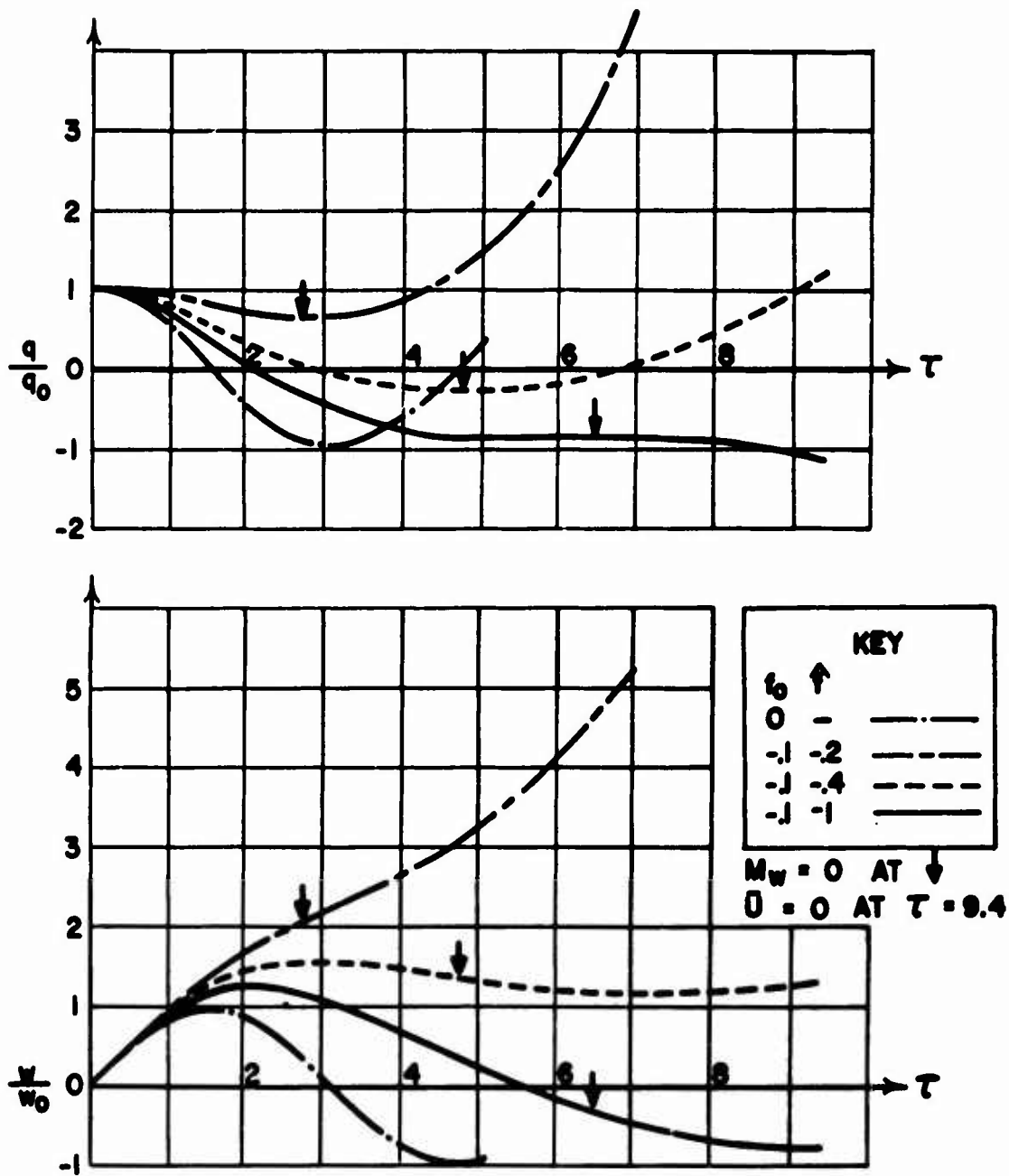
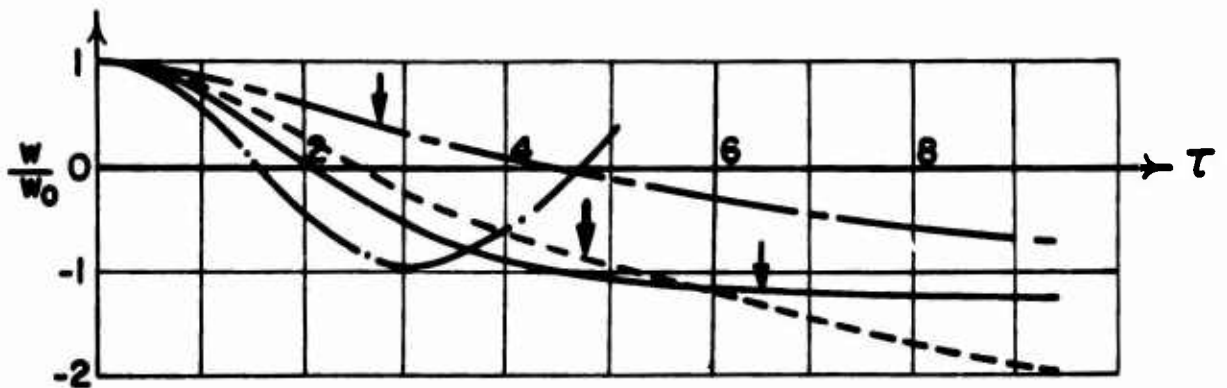
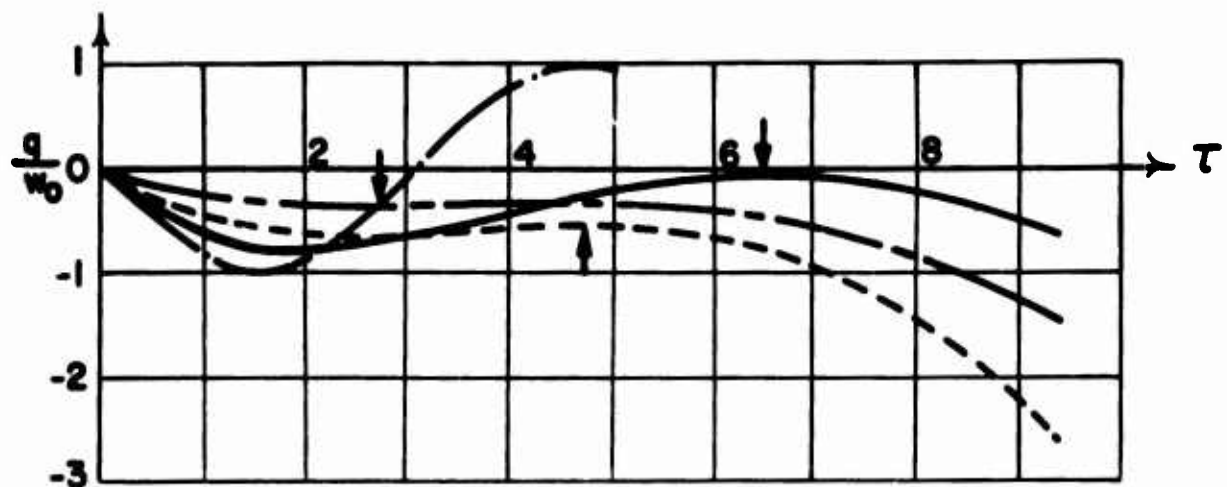


FIGURE 17b TRANSIENT RESPONSE FOR VARIOUS VALUES OF THE ACCELERATION PARAMETER, $\hat{\tau}$, WHEN $\hat{\tau}_0 = -1$

CONSTANT ACCELERATION

ATTITUDE STABILITY PROPORTIONAL TO FLIGHT SPEED AND NON-ZERO AT HOVER

NO AIRFRAME DAMPING



$M_w = 0$ AT \downarrow
 $\dot{w} = 0$ AT $\tau = 9.4$

| KEY | | |
|-------|-----------|-----------|
| f_0 | \hat{f} | |
| 0 | - | — · — · — |
| -1 | -2 | — · — · — |
| -1 | -4 | — · — · — |
| -1 | -1 | — · — · — |

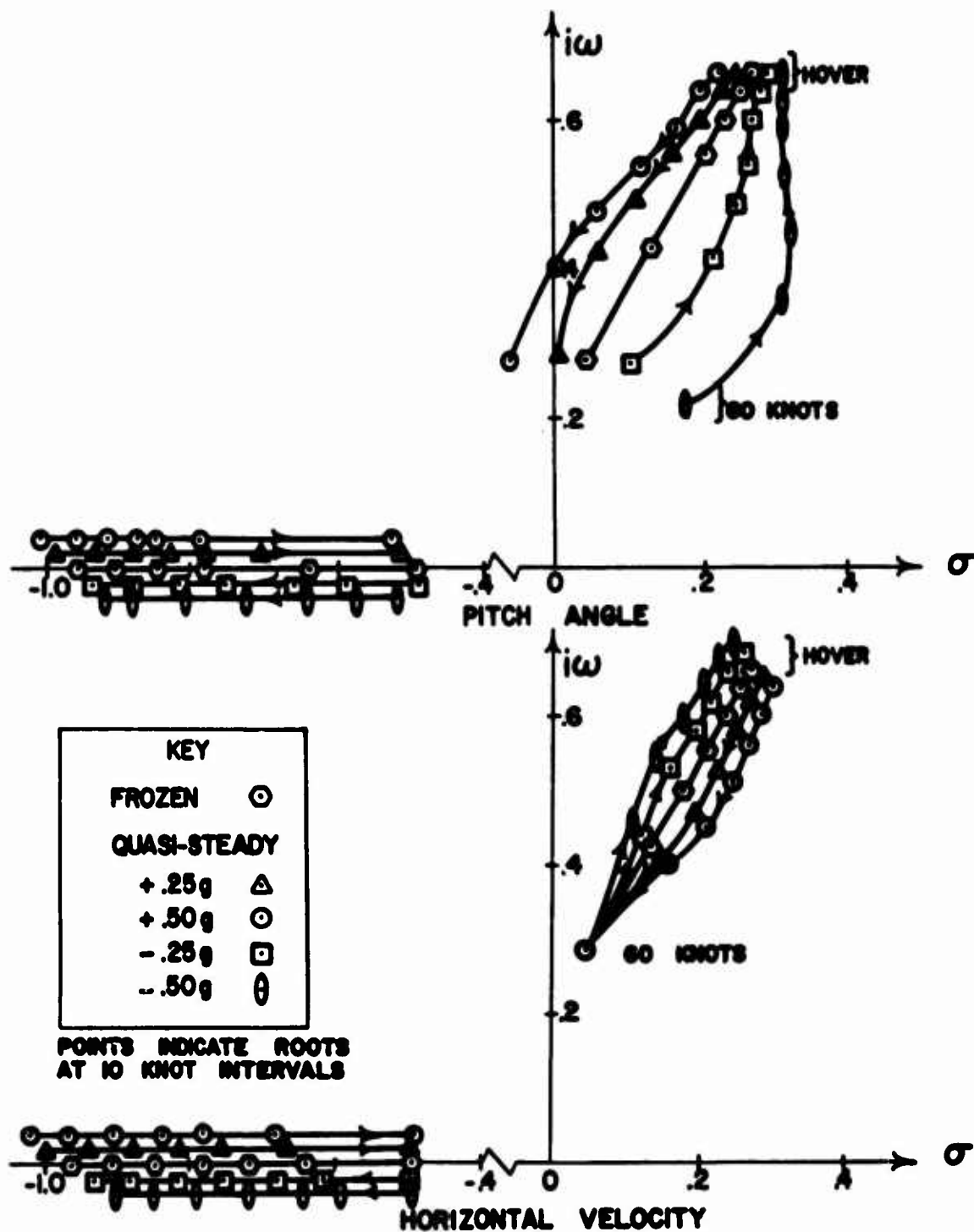
FIGURE 17c

TRANSIENT RESPONSE FOR VARIOUS VALUES OF THE ACCELERATION PARAMETER, \hat{f} , WHEN $f_0 = -1$

CONSTANT ACCELERATION

ATTITUDE STABILITY PROPORTIONAL TO FLIGHT SPEED AND NON-ZERO IN HOVER

NO AIRFRAME DAMPING



**FIGURE 18 FROZEN AND QUASI-STEADY ROOT LOCI
FOR VARIOUS CONSTANT ACCELERATION
TRANSITIONS**

TILT-WING VTOL

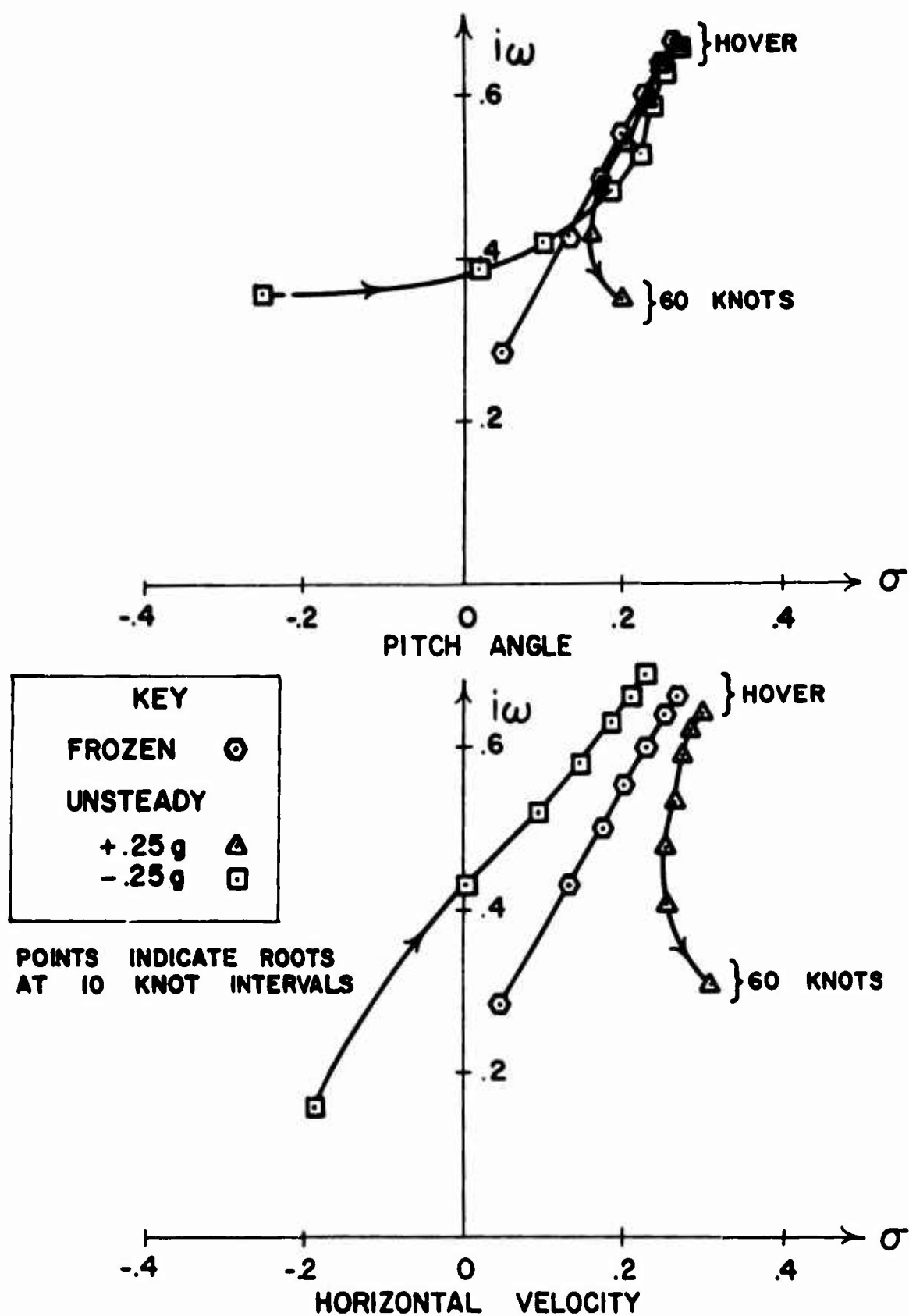


FIGURE 19 FROZEN AND UNSTEADY LOCI FOR CONSTANT
ACCELERATION TRANSITIONS
TILT-WING VTOL
ONLY OSCILLATORY MODE SHOWN

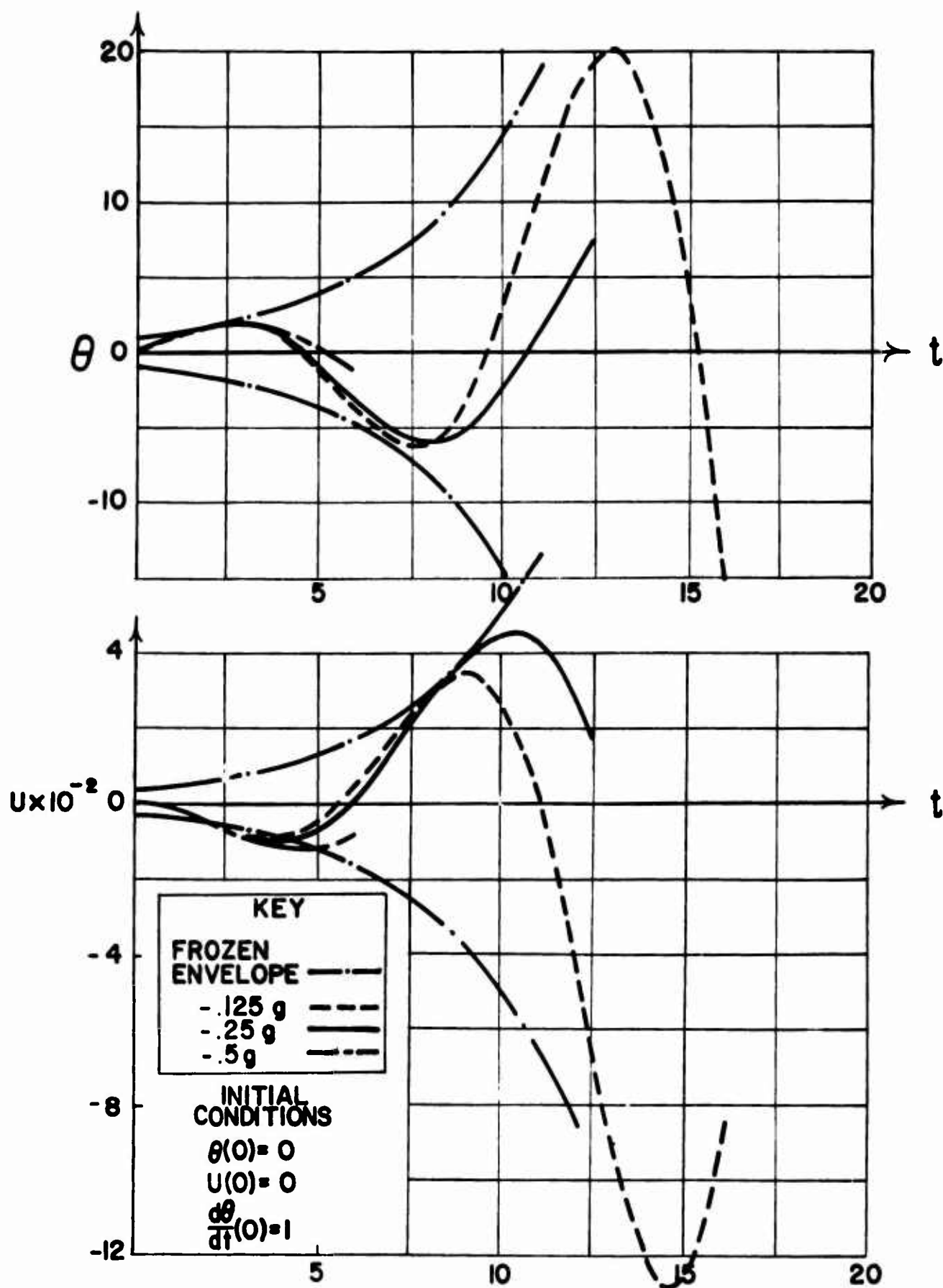


FIGURE 20a TRANSIENT RESPONSE OF TILT-WING VTOL DURING ACCELERATION TRANSITION FROM HOVER TO 60 KNOTS

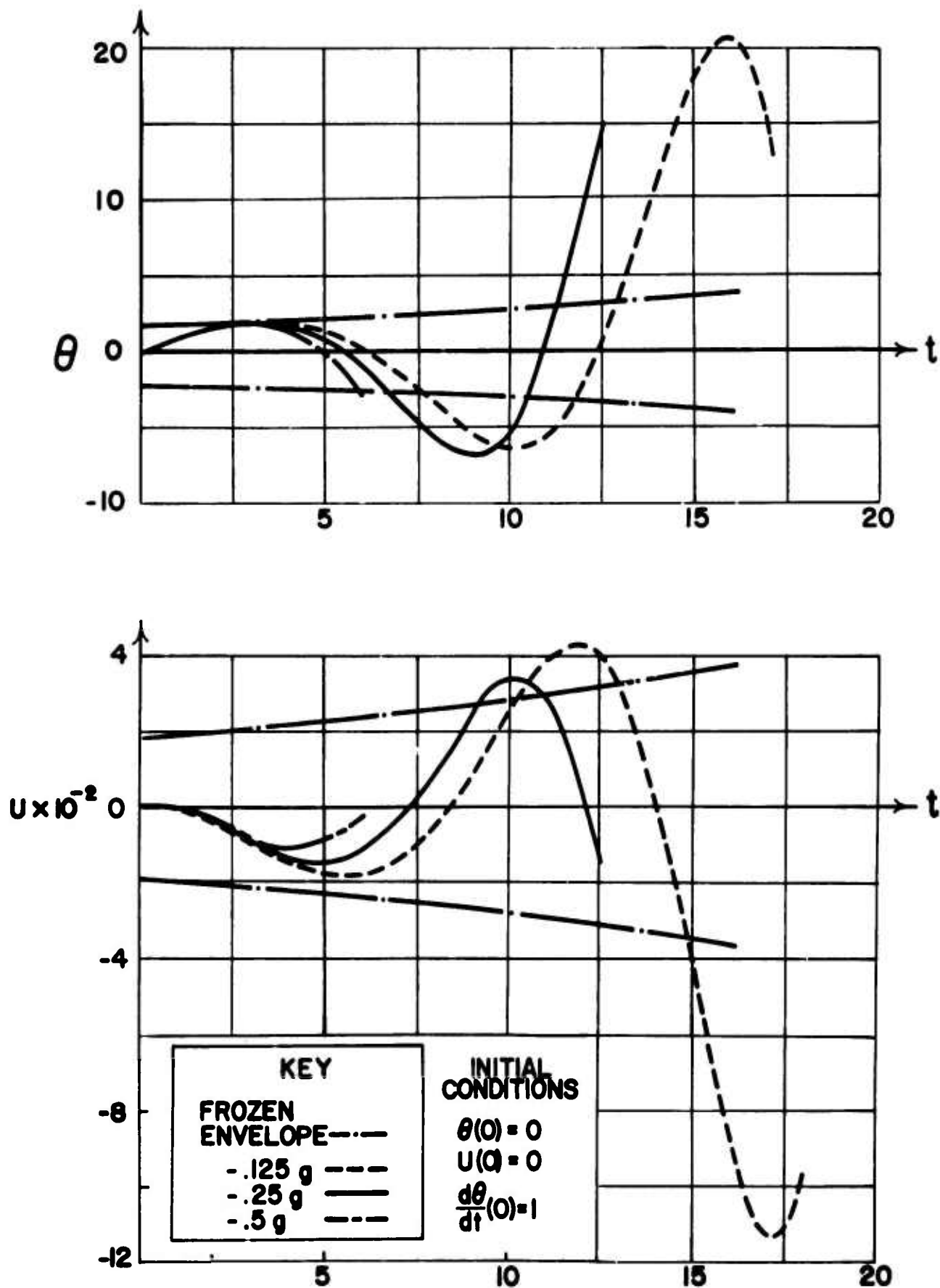


FIGURE 20b TRANSIENT RESPONSE OF TILT-WING VTOL DURING DECELERATING TRANSITION FROM 60 KNOTS TO HOVER

**FIGURE 21**

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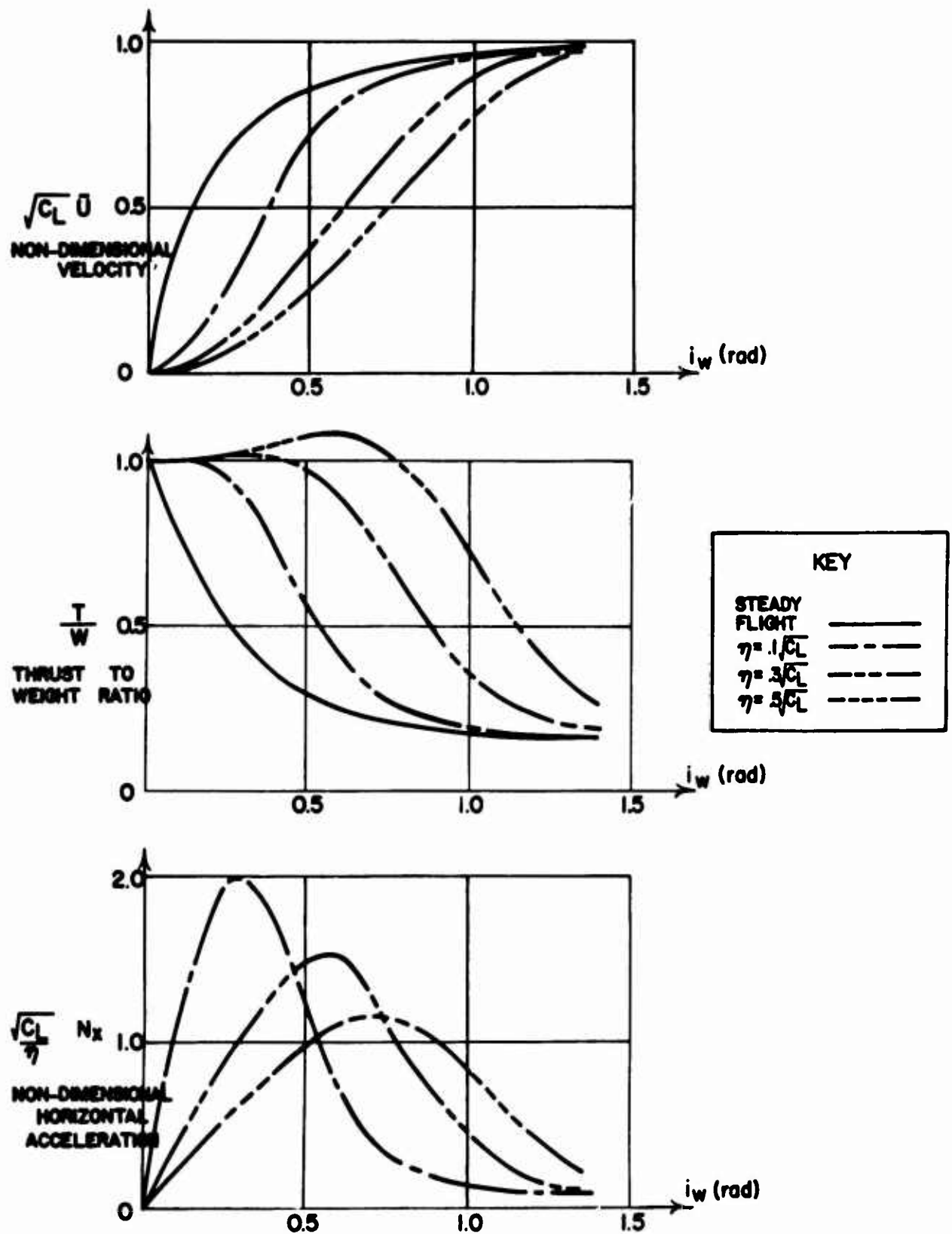


FIGURE 22 DEFLECTED JET VTOL
TRANSITION TIME HISTORIES, AS A FUNCTION
OF TILT RATE, FROM HOVER TO FORWARD FLIGHT

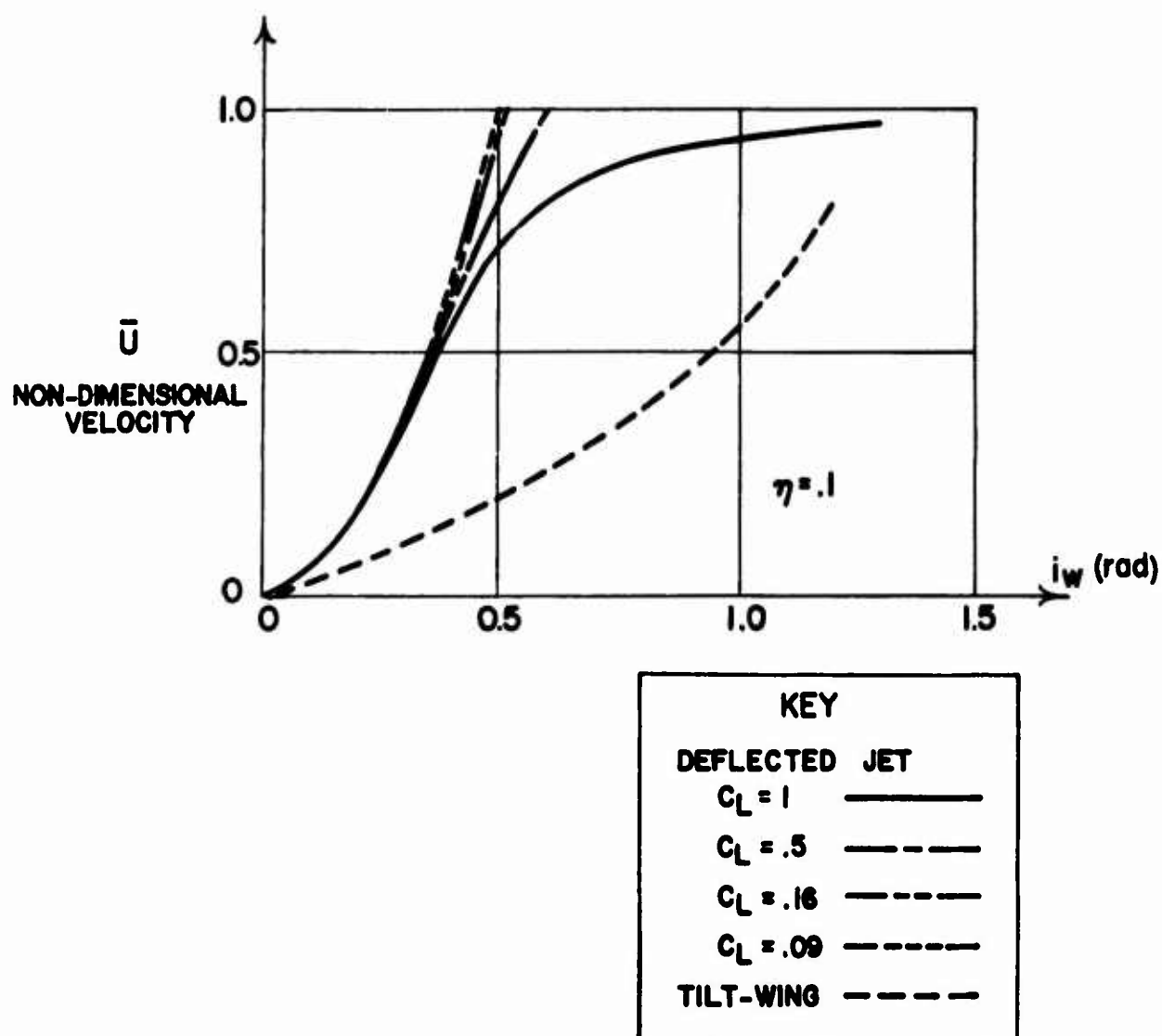


FIGURE 23 TRANSITION TIME HISTORIES OF A DEFLECTED JET VTOL, AS A FUNCTION OF LIFT COEFFICIENT, AND A TILT WING VTOL, AT A GIVEN TILT RATE

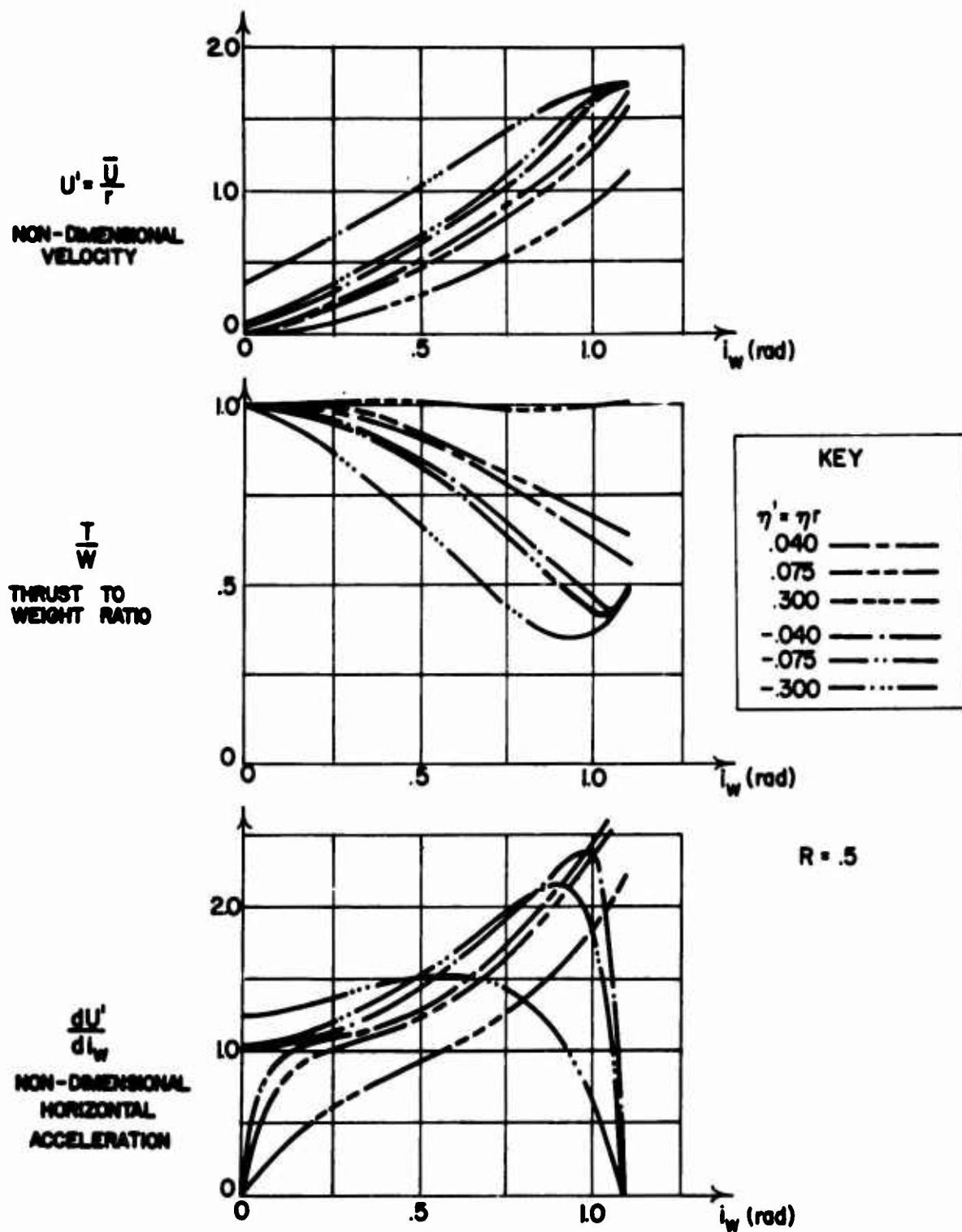
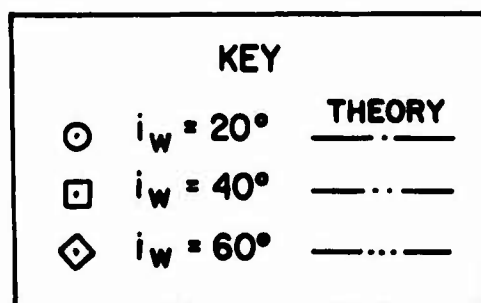
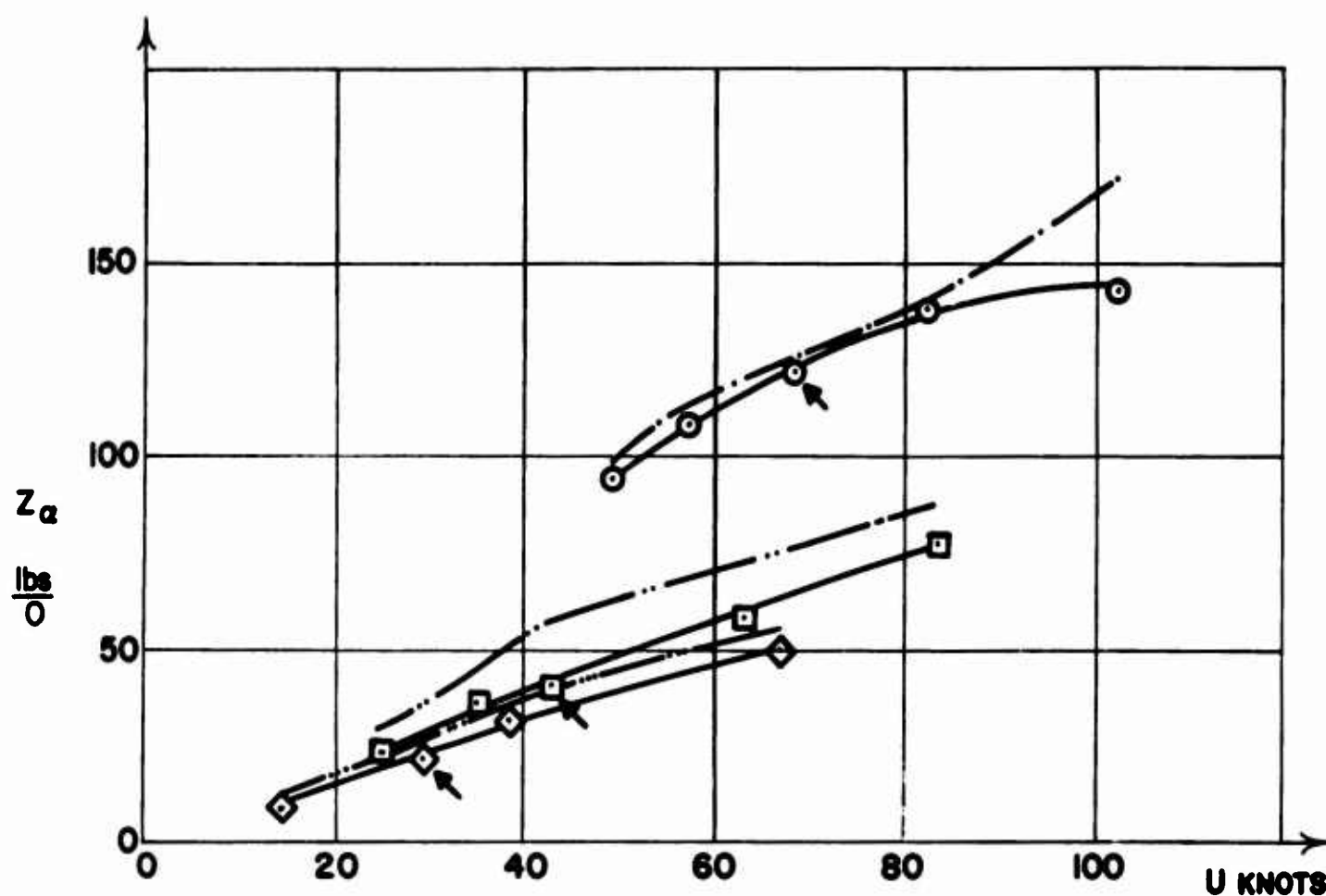


FIGURE 24 TILT WING VTOL
 TRANSITION TIME HISTORIES, AS A FUNCTION OF
 TILT RATE, FROM HOVER TO FORWARD FLIGHT
 AND FROM FORWARD FLIGHT TO HOVER



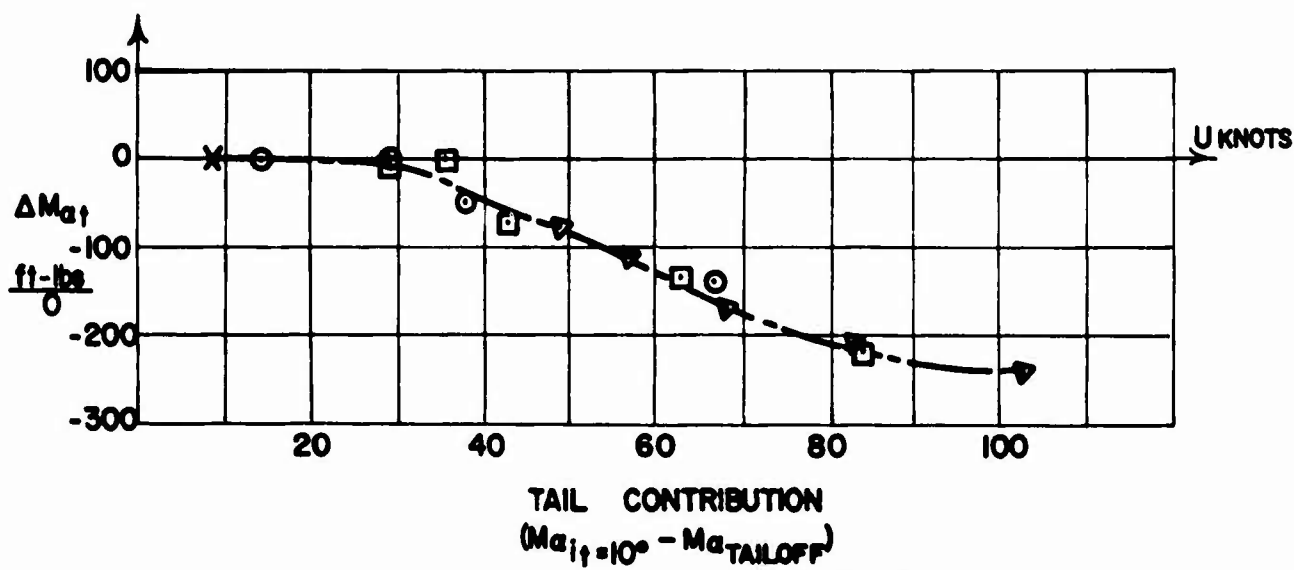
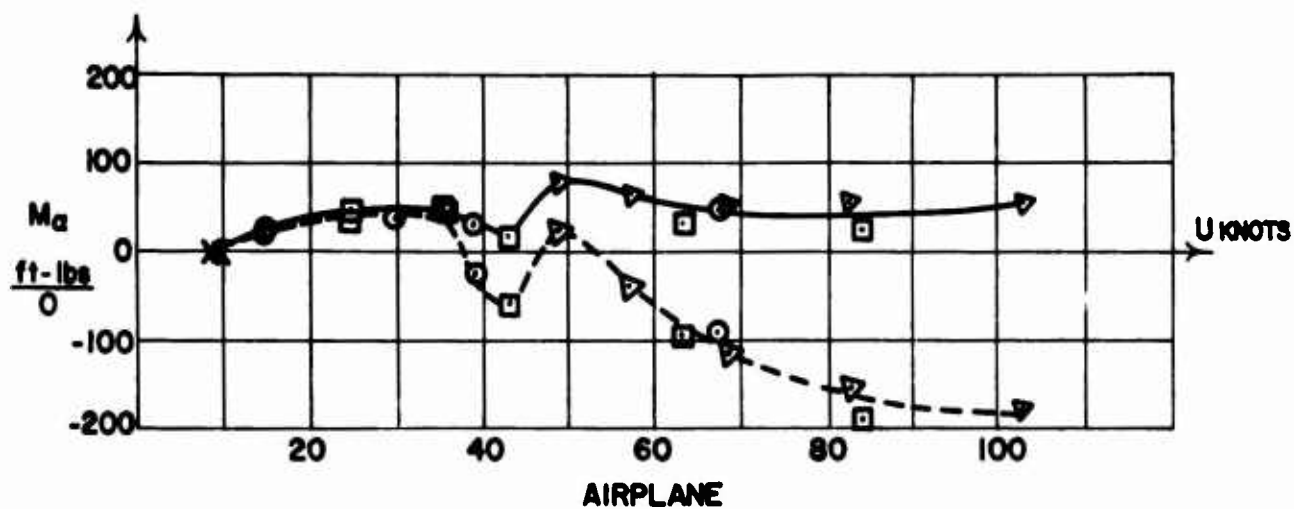
THEORY

$$Z_a \approx \frac{W}{\tan(i_w - \alpha_{OL})} - M\dot{U}$$

EXPERIMENTAL POINTS FOR TRIM FLIGHT (↑)
AND $\pm 0.25g$ INCREMENTS

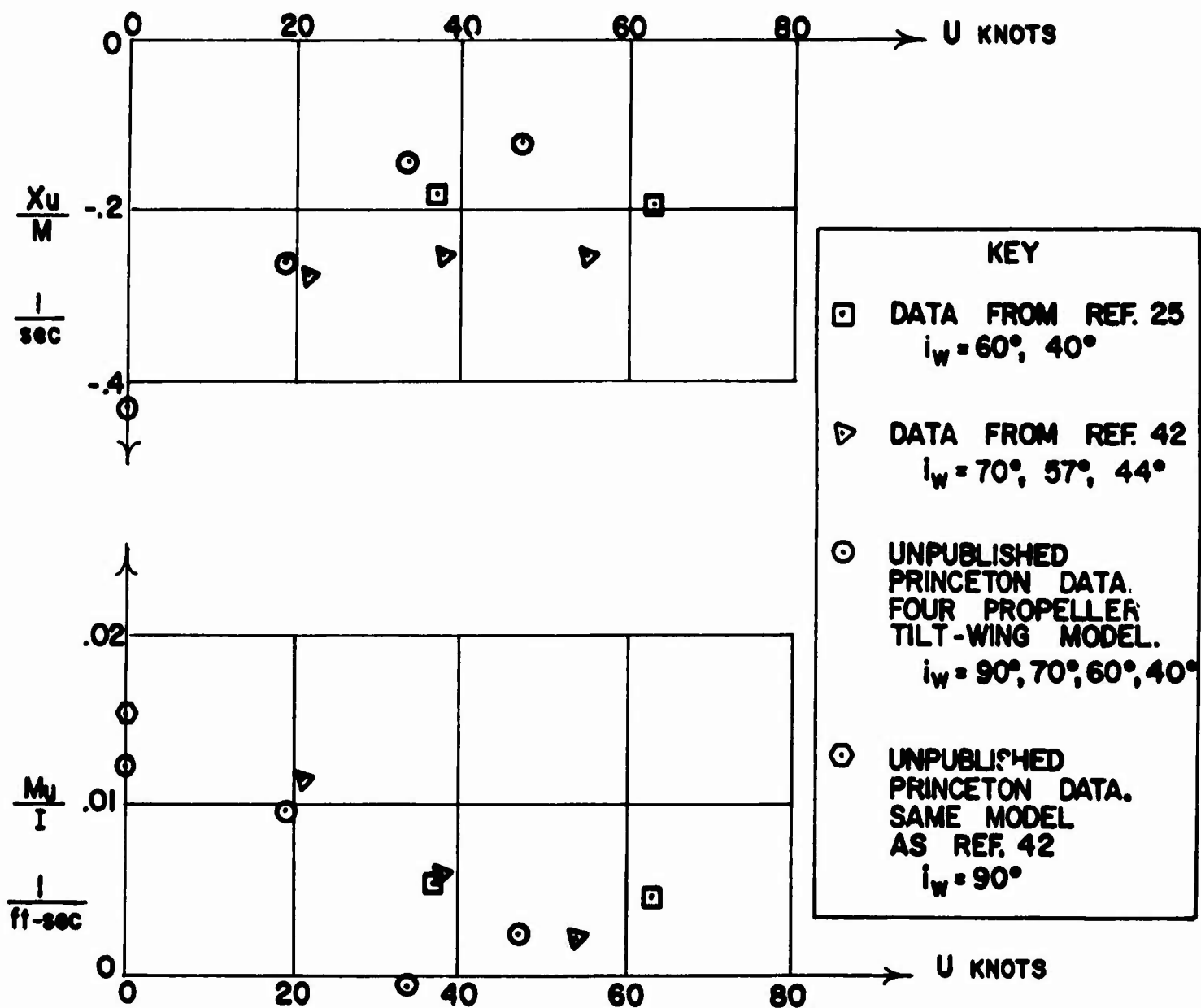
FIGURE 25 RATE OF CHANGE OF VERTICAL FORCE
WITH ANGLE OF ATTACK AS A FUNCTION
OF FLIGHT SPEED AND TILT ANGLE FOR
A TILT-WING VTOL

DATA FROM REFERENCE 25 AND COMPARISON
WITH THEORY



| KEY | |
|-----|-----------------------------|
| x | $i_w = 80^\circ$ |
| o | $i_w = 60^\circ$ |
| □ | $i_w = 40^\circ$ |
| ▽ | $i_w = 20^\circ$ |
| --- | TAIL ON $i_t = 10^\circ$ |
| — | TAIL OFF |

FIGURE 26 ATTITUDE STABILITY VARIATION WITH FLIGHT SPEED AND TILT ANGLE FOR TILT-WING VTOL
DATA FROM REFERENCE 25



NOTE:
ALL DATA IS SCALED TO AN EQUIVALENT
AIRPLANE WITH A GROSS WEIGHT OF
40,000 POUNDS ($I_y = 123,000$ SLUG-Feet²)

FIGURE 27 GENERAL TREND OF THE STABILITY DERIVATIVES X_u AND M_u AT LOW SPEEDS FOR TILT-WING AIRCRAFT

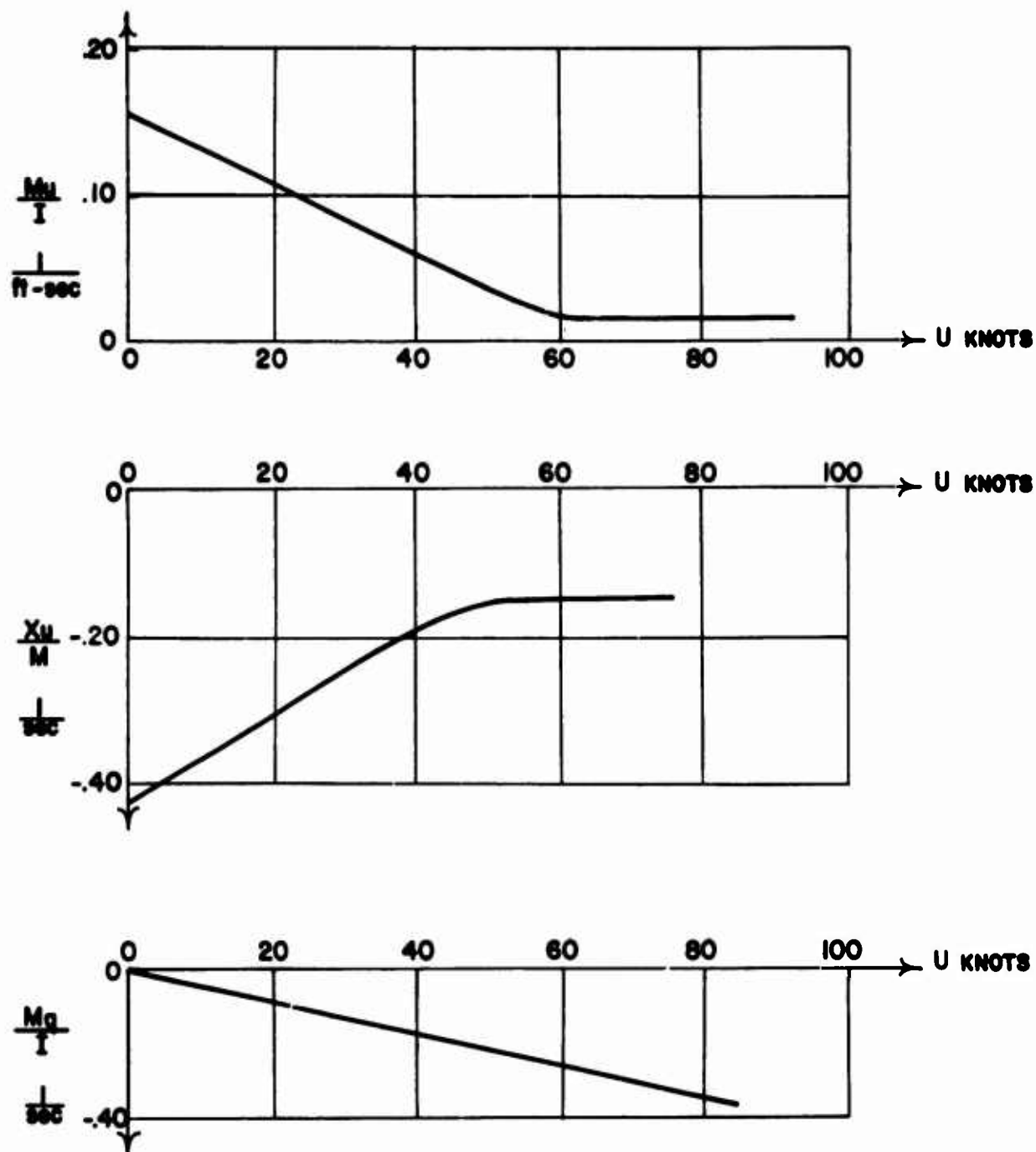


FIGURE 28 STABILITY DERIVATIVE VARIATIONS WITH FLIGHT SPEED USED FOR VTOL EXAMPLE IN CHAPTER V

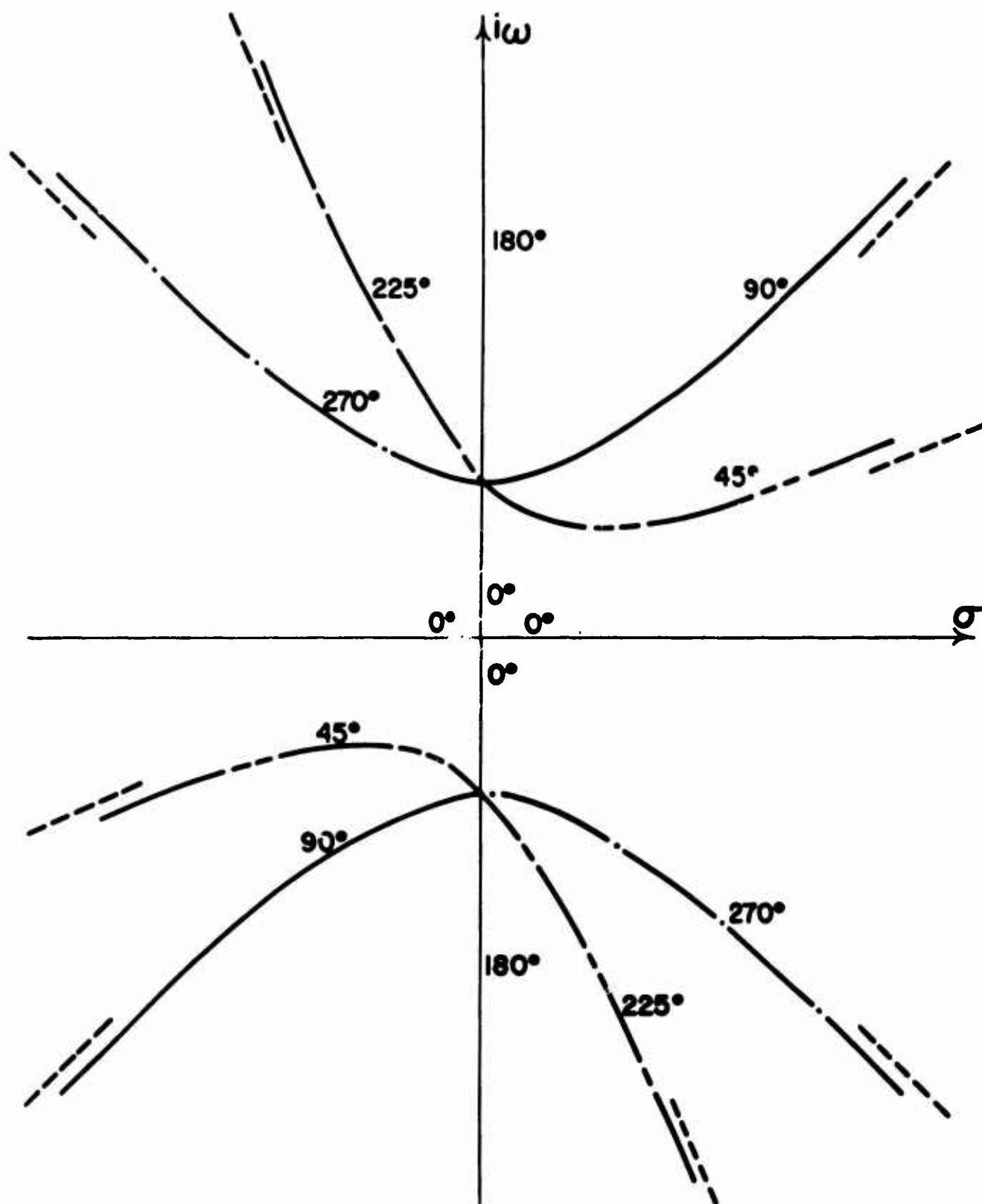


FIGURE 29 **VARIOUS ANGLE CONDITION LOCI**
FOR TWO COMPLEX POLES

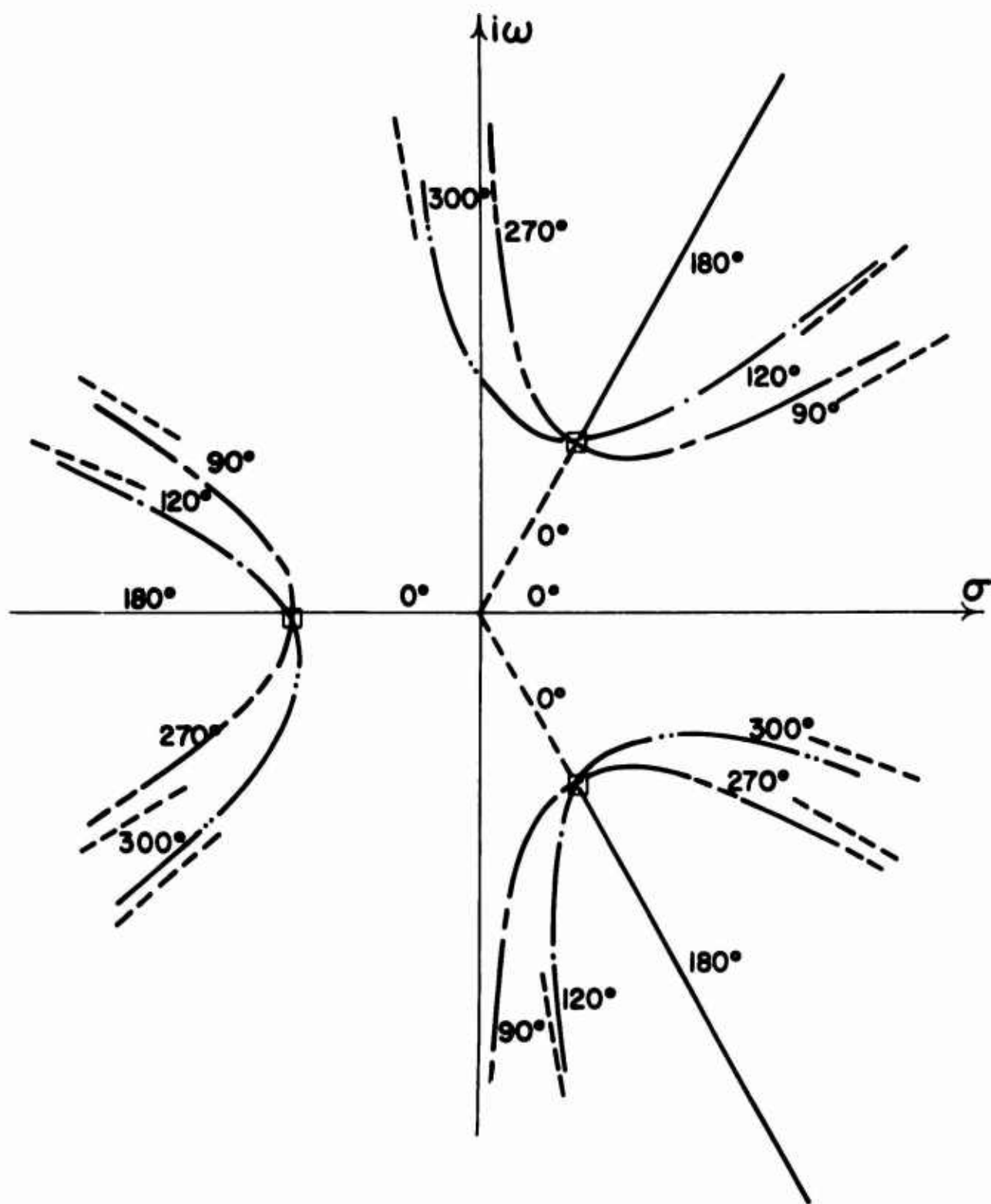


FIGURE 30 **VARIOUS ANGLE CONDITION LOCI FOR**
THREE POLES (ONE REAL POLE AND
A COMPLEX PAIR)

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| 13. ABSTRACT The dynamic response of conventional and VTOL aircraft with varying velocity is investigated. It is assumed that the dynamic motions of aircraft may be described by linear differential equations whose coefficients (stability derivatives) are functions of flight velocity, and therefore vary with time. Primary emphasis is placed on the evaluation of the general nature of the vehicle response and its departure from frozen system (constant coefficients) characteristics. An Approximate solution to linear differential equations with variable coefficients is presented, which, roughly speaking, applies if the percentage change of each of the characteristic roots per unit time is small compared to the spacing of the frozen roots on the complex plane. This approximate solution is interpreted in terms of a distortion of the frozen locus of roots on the complex plane. Variable coefficient effects may be rapidly estimated using this result. The properties of the solutions to linear equations with variable coefficients of significance to the problem are discussed. Of particular importance is the difference in the apparent damping of the various degrees of freedom in accelerated flight. The short period motion of aircraft is examined. For stability derivatives that vary linearly with velocity, the variations in the time histories of pitching velocity and vertical velocity appear as a distortion of the time scale of the frozen response. The transients in angle of attack and normal acceleration will exhibit different damping characteristics. For an aircraft that is stable in steady flight, the angle of attack response may be unstable in decelerated flight, and the normal acceleration response may be unstable in accelerated flight. The influence of other stability derivative variations on the short period motion and some examples of higher-order system dynamics are examined. | | | |

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